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RETAINING WALLS FOR EARTH

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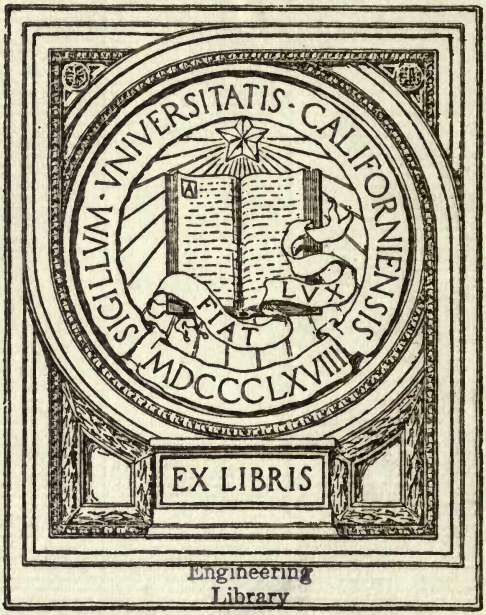

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RETAINING-WALLS FOR EARTH.

THE THEORY AS DEVELOPED BY PROF. JACOB J. WEYRAUCH.

EXPANDED AND SUPPLEMENTED BY PRACTICAL EXAMPLES, WITH NOTES ON LATER INVESTIGATIONS,

BY

MALVERD A. HOWE, C.E.

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JOHN WILEY AND SONS,
15 ASTOR PLACE.

1886.

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NOTE.

For the translation of Prof. Weyrauch's paper the writer is indebted to the labor of Prof. A. J. Du Bois, of the Sheffield Scientific School, Yale College, who had copies printed by the electric-pen process. However, only the leading equations of Prof. Weyrauch were given; hence a great deal of labor has been devoted to expanding, verifying, and filling in the intermediate steps of the work, and this nucleus of the mathematical part alone has grown to about double the original quantity. For most of the historical notes acknowledgment is also due to Prof. Du Bois for his article in *Journal of the Franklin Institute* for December, 1879, in which he called the attention of American engineers to the value of Weyrauch's theory.

M. A. H.

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PREFACE.

THE following theory of Prof. Weyrauch is presented for the use of the practical man, although, at the first glance, the array of mathematical formulæ may cause him to smile at such a statement. However, a brief examination of the results and their practical applications should convince him that a "long-felt want" has been supplied, not only in the graphical constructions that can be performed anywhere with the simplest instruments, but by formulæ so simple that substitutions can be made and solutions obtained in a very short time.

The mathematical operations are somewhat tedious, and are now presented for the first time in English, fully expanded and verified, so as to be easily followed by those who are inclined to doubt the results or wish to satisfy themselves that they are correct.

In fact, all that it is absolutely necessary for the practical man to have is the contents of the *Recapitulation of Formulæ*, in order to determine the earth-thrust and its direction for a wall not leaning backward.

For walls leaning backward, Prof. Rankine's method has been combined with Weyrauch's.

Walls having a curved profile and those with counterforts have not met with the approval of American engin-

eers, and with good reason, as they can as yet be treated only by empirical formulæ; for these reasons they will not be considered.

Some may question the accuracy of the theory for surface of earth inclined, i.e., for surcharged wall; yet in any case Prof. Weyrauch's theory is to be preferred to any of those previously advanced.

An attempt has been made to present the subject in a simple manner, and to show by a few examples the simplicity of the application of the formulæ and constructions.

The reader who does not care to follow the theory until he is persuaded of its practical value in application should turn at once to examples and discussions in Part II.

THAYER SCHOOL OF CIVIL ENGINEERING,
April, 1886.

M. A. H.

INTRODUCTION.

OLDER THEORIES.

RETAINING-WALLS were first treated in 1687, but until Coulomb's time no theories were advanced that are worthy of much notice, as they were for the most part founded upon mere assumptions, for which the reasons, if given at all, were statements unproved.

In 1773 Coulomb founded a new school that assumed the earth-pressure to act normally to the wall, and to be induced by a prism of maximum thrust.

As Coulomb's theory was the only one generally accepted for some sixty or more years, a brief outline of the principal points will be given.

According to Coulomb, the surface of rupture is a plane along which a prism of rupture tends to slide; he also assumes the direction of the earth's thrust to be normal to the wall.

The weight of the prism of thrust, G, and the reaction of the wall, E, are decomposed into forces respectively perpendicular and parallel to the surface of rupture. Then the difference of the horizontal components must represent the resistance to sliding of the prism of thrust. This resistance consists of *friction*, which is proportional to the normal pressure, and *cohesion*, which is proportional to the surface of sliding.

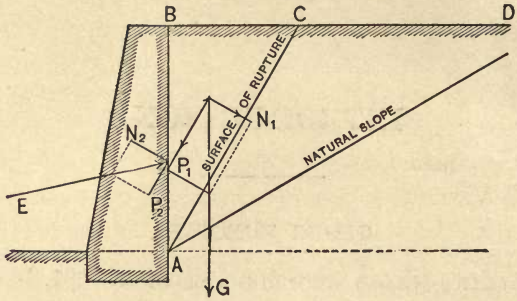

FIG. 0.

Let f = the coefficient of friction.
c = the coefficient of cohesion.
γ = the weight of a unit of volume of the earth.
h = the height of the wall.
x = the distance BC.
E = the earth-thrust against the wall.

The wall will be assumed as vertical, and the earth-surface as horizontal (as this is the only case discussed by Coulomb).

Then from the above figure

$$E = \frac{\gamma hx(h - fx) - 2c(h^2 + x^2)}{2(x + fh)}. \quad . \quad . \quad (A)$$

Coulomb then reasons that there must be somewhere a surface of rupture corresponding to a prism that will exert a maximum pressure against the wall; and he proceeds to differentiate the above expression with respect to x, and finds that E is a maximum when $x = -fh + h\sqrt{1+f^2}$,

an expression wholly independent of the coefficient of cohesion.

The point of application of E is shown to be at one third the height of the wall.

These deductions are correct for this particular case, as Prof. Weyrauch's theory shows. But when either the earth or the wall is inclined, the direction of E is not normal to the wall, but makes an angle, δ, with the normal.

The value of E, as given by Coulomb, depends directly upon the position of the surface of rupture, and changes in intensity but not in direction as the surface of rupture is assumed to change. In reality E is constant for any given wall and earth, and does not depend upon the position of the surface of rupture. See Recapitulation; and notice that ω, the angle the surface of rupture makes with the vertical, does not occur in the equations for the value of E.

Later writers have proved (?) Coulomb's prism of maximum thrust to be limited by a plane which bisects the angle made by the natural slope of the earth and the vertical rear face of the wall.

In the nineteenth century, Rankine, Levy, and Mohr have considered the conditions of the earth-particles, and arrive at their results by integration.

Rebhahn (1871) and Winkler (1872) advanced theories founded on assumptions identical with those of Rankine.

Rankine assumed the surface of rupture to be a plane, and that the direction of the earth's thrust is parallel to the top surface.

As has been said before, Coulomb assumed the pressure

of the earth to act normally to the wall. Since 1840 it has been customary to assume the earth-pressure to make an angle with the normal equal to the angle of repose. If this were true, a horizontal wall would be pressed, not vertically, but by forces acting at an angle with the vertical equal to the angle of repose, which is manifestly incorrect.

It will be seen that Prof. Weyrauch's theory is closely allied to Prof. Rankine's, but conclusively proves that the earth-pressure acts parallel to the top surface of the earth only in special cases.

WEYRAUCH'S

THEORY OF THE RETAINING-WALL.*

PART FIRST.

In the following the earth is supposed without cohesion, and its pressure is determined independently of any arbitrary assumptions as to direction of the earth-pressure, and with sole reference to the three necessary conditions of equilibrium. The single and only supposition, then, is as follows: *That the forces upon any imaginary plane-section through the mass of earth have the same direction.*

This assumption lies at the foundation of *all* theories of earth-pressure against retaining-walls. For those cases, therefore, to which the following discussion does not apply no complete or satisfactory theory is yet possible. In what follows, the ordinary assumption as to the direction of the earth-pressure will be proved to be *incorrect,* except for special cases.

---

* *Zeitschrift für Baukunde,* Band I. Heft 2, 1878.

I.

GENERAL RELATIONS.

Let the surface of the earth have any form, and the wall AB, Fig. 1, have any inclination. The earth-pressure makes any angle, δ, with the normal to the wall.

Suppose through the point A the plane AC. Then the weight G of the prism ABC is held in equilibrium by the

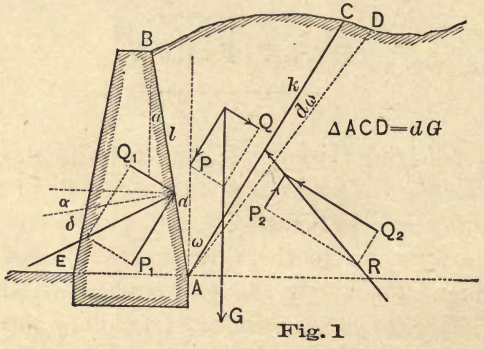

Fig. 1

FIG. 1.

reaction of the wall, E, and by the resultant, R, of all the forces acting upon AC.

Now decompose E, G, and R into components parallel and normal to AC; then for every unit in length of the wall, denoting by e, g, and r the lever-arms of E, G, and R respectively with reference to A, the sum of the forces parallel to $AC = 0$, or

$$P - P_1 - P_2 = 0; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the sum of the forces perpendicular to $AC = 0$, or

$$Q + Q_1 - Q_2 = 0; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the sum of moments about $A = 0$, or

$$Gg + Ee - Rr = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equation (3) was first introduced by Prof. Weyrauch.

Further, according to the theory of friction, if φ is the coefficient of friction for earth on earth,

$$\frac{P_2}{Q_2} \angle = \tan \varphi \text{ or } \frac{P - P_1}{Q + Q_1} \angle = \tan \varphi. \quad . \quad . \quad (4)$$

If now there is any plane for which

$$P - P_1 = (Q + Q_1) \tan \varphi, \quad . \quad . \quad . \quad (5)$$

this plane AC will be a plane of equilibrium, and $\frac{P - P_1}{Q + Q_1}$ will be a maximum, or

$$\frac{d\left(\frac{P - P_1}{Q + Q_1}\right)}{d\omega} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This plane is designated as the "surface of rupture."

From Fig. 1, for every position of AC,

$$P = G \cos \omega, \qquad Q = G \sin \omega,$$
$$P_1 = E \sin (\omega + \alpha + \delta), \qquad Q_1 = E \cos (\omega + \alpha + \delta).$$

Substituting the above values of P, P_1, Q, and Q_1 in equation (5), it becomes

$$G \cos \omega - E \sin (\omega + \alpha + \delta) = [G \sin \omega + E \cos (\omega + \alpha + \delta)] \tan \varphi;$$

and when ω refers to the surface of rupture, the earth-pressure upon AB becomes

$$E = \frac{\cos \omega - \sin \omega \tan \varphi}{\sin (\omega + \alpha + \delta) + \cos (\omega + \alpha + \delta) \tan \varphi} G.$$

Substituting the value of $\tan \varphi$ or $\frac{\sin \varphi}{\cos \varphi}$, this becomes

$$E = \frac{\cos \varphi \cos \omega - \sin \omega \sin \varphi}{\sin (\omega + \alpha + \delta) \cos \varphi + \cos (\omega + \alpha + \delta) \sin \varphi} G,$$

which becomes

$$E = \frac{\cos (\varphi + \omega)}{\sin (\varphi + \omega + \alpha + \delta)} G. \quad . \quad . \quad (7)$$

In order to refer to the surface of rupture, the following relation must exist:

$$\frac{d\left(\frac{G \cos \omega - E \sin (\omega + \alpha + \delta)}{G \sin \omega + E \cos (\omega + \alpha + \delta)}\right)}{d\omega} = 0. \quad (7a)$$

Performing the differentiation indicated in the equation (7a), considering G and ω as the variables, it becomes

$$\frac{\begin{array}{l} + [dG \cos \omega - \sin \omega d\omega \, G - E \cos (\omega + \alpha + \delta) d\omega] \, [G \sin \omega + E \cos (\omega + \alpha + \delta)] \\ - [dG \sin \omega + \cos \omega d\omega \, G - E \sin (\omega + \alpha + \delta) d\omega] \, [G \cos \omega - E \sin (\omega + \alpha + \delta)] \end{array}}{[G \sin \omega + E \cos (\omega + \alpha + \delta)]^2 d\omega} =$$

$$= 0; \quad . \quad . \quad .

or

$$+\frac{dG\cos\omega}{d\omega}[G\sin\omega+E\cos(\omega+\alpha+\delta)]-[G\sin\omega+E\cos(\omega+\alpha+\delta)]^2$$
$$-\frac{dG\sin\omega}{d\omega}[G\cos\omega-E\sin(\omega+\alpha+\delta)]-[G\cos\omega-E\sin(\omega+\alpha+\delta)]^2$$
$$\overline{[G\sin\omega+E\cos(\omega+\alpha+\delta)]^2}=$$
$$=0. \quad (7d$$

Now, since

$$\cos\omega\cos(\omega+\alpha+\delta)+\sin\omega\sin(\omega+\alpha+\delta)=\cos(\alpha+\delta)$$
and $$\sin^2\omega+\cos^2\omega=1,$$

by clearing of fractions this becomes

$$-\frac{EdG\cos(\alpha+\delta)}{d\omega}+G^2-2GE\sin(\alpha+\delta)+E^2=0. \quad (7e)$$

Now since $dG=\frac{1}{2}k\ .\ d\omega\ .\ k\gamma$, equation (7e) reduces to

$$G^2-2GE\sin(\alpha+\delta)-\frac{Ek^2\gamma\cos(\alpha+\delta)}{2}+E^2=0, \quad (7f)$$

which becomes, after dividing by GE,

$$\frac{G}{E}-2\sin(\alpha+\delta)-\frac{k^2\gamma\cos(\alpha+\delta)}{2G}+\frac{E}{G}=0. \quad (8)$$

Substituting the value of $\frac{E}{G}$ from equation (7), transposing and multiplying by two, equation (8) reduces to

$$\frac{2\sin(\phi+\alpha+\omega+\delta)}{\cos(\phi+\omega)}-4\sin(\alpha+\delta)+\frac{2\cos(\phi+\omega)}{\sin(\phi+\omega+\alpha+\delta)}=\frac{k^2\gamma\cos(\alpha+\delta)}{G}, \quad (8a)$$

whence

$$G = \frac{k^2\gamma \cos(\alpha+\delta)}{\frac{2\sin(\phi+\omega+\alpha+\delta)}{\cos(\phi+\omega)} - 4\sin(\alpha+\delta) + \frac{2\cos(\phi+\omega)}{\sin(\phi+\omega+\alpha+\delta)}}, \quad \ldots\ldots (8b)$$

which reduces to

$$G = \frac{\cos(\phi+\omega)\sin(\phi+\omega+\alpha+\delta)\cos(\alpha+\delta)k^2\gamma}{2[\sin^2(\phi+\omega+\alpha+\delta) - 2\sin(\alpha+\delta)\cos(\phi+\omega)\sin(\phi+\omega+\alpha+\delta) + \cos^2(\phi+\omega)]}. \quad (8c)$$

Since

$$\sin(\varphi+\omega+\alpha+\delta) = \sin(\varphi+\omega)\cos(\alpha+\delta) + \cos(\varphi+\omega)\sin(\alpha+\delta),$$

the parenthetical portion of the denominator becomes

$$\begin{aligned}
&\sin^2(\varphi+\omega)\cos^2(\alpha+\delta)\\
&\quad + 2\sin(\alpha+\delta)\cos(\varphi+\omega)\sin(\varphi+\omega)\cos(\alpha+\delta)\\
&\quad + \cos^2(\varphi+\omega)\sin^2(\alpha+\delta)\\
&\quad - 2\sin(\alpha+\delta)\cos(\varphi+\omega)\sin(\varphi+\omega)\cos(\alpha+\delta)\\
&\quad - 2\sin(\alpha+\delta)\cos(\varphi+\omega)\cos(\varphi+\omega)\sin(\alpha+\delta)\\
&\quad + \cos^2(\varphi+\omega),
\end{aligned}$$

or

$$\begin{aligned}
&\sin^2(\varphi+\omega)\cos^2(\alpha+\delta)\\
&\quad - 2\sin^2(\alpha+\delta)\cos^2(\varphi+\omega)\\
&\quad + \sin^2(\alpha+\delta)\cos^2(\varphi+\omega) + \cos^2(\varphi+\omega),
\end{aligned}$$

or $$\sin^2(\varphi+\omega)\cos^2(\alpha+\delta) - \cos^2(\varphi+\omega)\sin^2(\alpha+\delta) + \cos^2(\varphi+\omega),$$

or $$\sin^2(\varphi+\omega)\cos^2(\alpha+\delta) + \cos^2(\varphi+\omega)[1-\sin^2(\alpha+\delta)],$$

or $$\sin^2(\varphi+\omega)\cos^2(\alpha+\delta) + \cos^2(\varphi+\omega)\cos^2(\alpha+\delta),$$

or $$\cos^2(\alpha+\delta)[\sin^2(\varphi+\omega) + \cos^2(\varphi+\omega)],$$

which equals $\cos^2(\alpha+\delta)$, and equation (8*c*) becomes, after dividing by $\cos(\alpha+\delta)$ and factoring,

$$G = \frac{\cos(\varphi+\omega)\sin(\varphi+\omega+\alpha+\delta)}{\cos(\alpha+\delta)} \cdot \frac{k^2\gamma}{2} = \text{Function } \gamma, \quad (9)$$

from which

$$\sin(\varphi+\omega+\alpha+\delta) = \frac{2G}{k^2\gamma} \cdot \frac{\cos(\alpha+\delta)}{\cos(\varphi+\omega)},$$

which being substituted in equation (7) gives

$$E = \frac{G\cos(\varphi+\omega)}{\frac{2G\cos(\alpha+\delta)}{k^2\gamma\cos(\varphi+\omega)}} = \frac{\cos^2(\varphi+\omega)}{\cos(\alpha+\delta)} \cdot \frac{k^2\gamma}{2}. \quad (10)$$

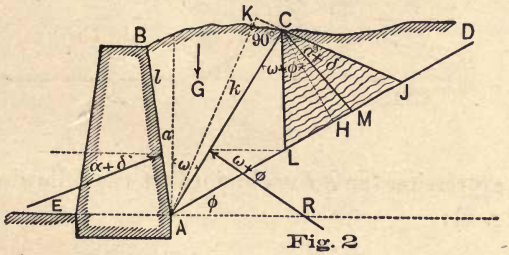

Fig. 2.

FIG. 2.

And, since the sum of the horizontal components of E, G, and R must be equal to 0, or Fig. 2,

$$E\cos(\alpha+\delta) = R\cos(\omega+\varphi),$$

and

$$R = E\frac{\cos(\alpha+\delta)}{\cos(\omega+\varphi)};$$

which becomes, after substituting the value of E from equation (10),

$$R = \cos(\varphi+\omega)\frac{k^2\gamma}{2}. \quad . \quad . \quad . \quad . \quad (11)$$

Let AD, Fig. 2, be the natural slope of the ground. From C let fall the perpendicular CH, and draw CJ, making the angle $(\alpha+\delta)$ with CH; then

$$CH = k\cos(\varphi+\omega), \qquad AJ = \frac{\sin(\varphi+\omega+\alpha+\delta)}{\cos(\alpha+\delta)}k.$$

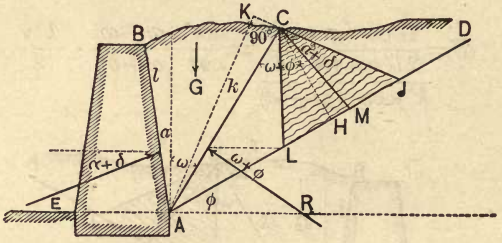

FIG. 2.

The expression for AJ is obtained in the following manner (Fig. 2):

$$CH = k\cos(\varphi+\omega), \quad AH = k\sin(\varphi+\omega),$$
$$HJ : CH :: \sin(\alpha+\delta) : \cos(\alpha+\delta),$$

and $$HJ = \frac{CH\sin(\alpha+\delta)}{\cos(\alpha+\delta)} = \frac{\cos(\varphi+\omega)\sin(\alpha+\delta)}{\cos(\alpha+\delta)}k,$$

$$AH + HJ = AJ$$
$$= \frac{\sin(\varphi+\omega)\cos(\alpha+\delta)+\cos(\varphi+\omega)\sin(\alpha+\delta)}{\cos(\alpha+\delta)}k,$$

which reduces to

$$AJ = \frac{\sin(\varphi+\omega+\alpha+\delta)}{\cos(\alpha+\delta)}k;$$

and hence, according to equation (9),

$$G = \text{Func. } \gamma = \gamma \Delta ACJ. \quad . \quad . \quad . \quad . \quad (12)$$

Also, if AK is perpendicular to CJ,

$$\frac{CH}{AK} = \frac{k\cos(\varphi+\omega)}{k\sin(\varphi+\omega+\alpha+\delta)} = \frac{E}{G};$$

and if JL is made equal to JC, then, since the perpendicular from L upon CJ is equal to CH,

$$\frac{\Delta CJL}{\Delta CJA} = \frac{CH}{AK} = \frac{E}{G},$$

or

$$E = \gamma \Delta CJL. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If, finally, $AM = AC$,

$$\Delta ACM = \frac{AM \,.\, CH}{2} = \tfrac{1}{2}k^2\cos(\varphi+\omega),$$

or

$$R = \gamma \Delta ACM. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

All these geometrical results may be summed up as follows :

Draw from the highest point C of the surface of rupture a line CJ, which makes with the normal CH to the natural slope the angle $\alpha + \delta$, or the angle which the earth-pressure makes with the horizontal ; then the ΔACJ is

equal in area to the ΔABC, the prism of rupture. Then lay off $JL = JC$ and $AM = AC$ and draw CL and CM; then for every unit in length of the wall the following relations exist:

$$\left.\begin{array}{ll} \text{Weight of prism of rupture,} & G = \gamma \Delta CAJ; \\ \text{Earth-pressure upon wall,} & E = \gamma \Delta CJL; \\ \text{Reaction of the surface of rupture,} & R = \gamma \Delta CAM. \end{array}\right\} \quad (14a)$$

The first two relations were first made known by Rebhahn in 1871, for $\delta = 0$ or φ.

$$\text{Since, now, } G : E : R = AJ : JC : CA, \quad . \quad . \quad . \quad (15)$$

it can be asserted that—

The weight of the prism of rupture and the reactions of the wall and of the surface of rupture are to each other as the three sides of the ΔACJ.

Thus far no assumption whatever has been made as to the value of the angle δ. This is determined by equation (3), which, in all theories following Coulomb's method, does not occur.

II.

PLANE EARTH-SURFACE INCLINED.

ADOPT in this case the notation of Fig. 3, and let E be first determined for any value of δ.

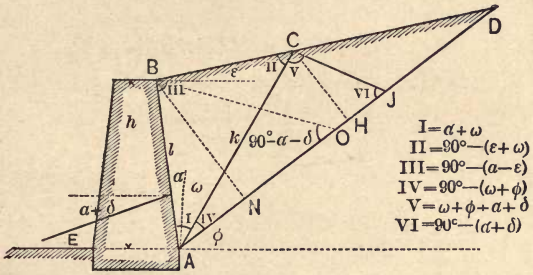

FIG. 3.

If AC is the surface of rupture, then $\Delta ABC = \Delta ACJ$; or, since

$$\frac{AB}{AC} = \frac{\sin\ II}{\sin\ III}, \qquad AB = AC\frac{\sin\ II}{\sin\ III}.$$

In like manner, $$AJ = AC\frac{\sin\ V}{\sin\ VI}.$$

But since $$\Delta ABC = \Delta ACJ,$$

$$AB \,.\, AC \sin I = AJ \,.\, AC \sin IV; \quad . \quad . \quad (16)$$

or $$\frac{\sin I \sin II}{\sin III} = \frac{\sin IV \sin V}{\sin VI}; \quad . \quad . \quad . \quad (16a)$$

or, finally,

$$\sin(\alpha+\omega)\cos(\varepsilon+\omega)\cos(\alpha+\delta) = \sin(\varphi+\omega+\alpha+\delta)\cos(\varphi+\omega)\cos(\alpha-\varepsilon). \quad (16b)$$

Further, from Fig. 3, if BN is perpendicular to AD,

$$\Delta ADB = 2\Delta AJC + \Delta JDC,$$

or

$$AD \,.\, BN = 2AJ \,.\, CH + JD \,.\, CH;$$

and since

$$\frac{BN}{CH} = \frac{BO}{CJ} = \frac{OD}{JD},$$

and

$$AD \,.\, OD = JD\,(AJ + AD),$$

$$AD\,(AD - AO) = (AD - AJ)\,(AJ + AD),$$

whence

$$AO \,.\, AJ = AJ \,.\, AD. \quad . \quad . \quad . \quad (17)$$

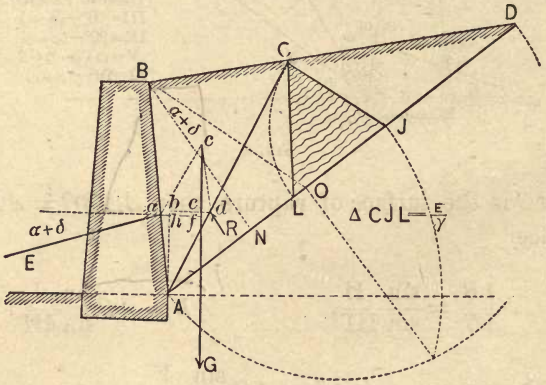

Fig. 3′.

Upon this relation rests the well-known construction of Poncelet for the earth-pressure. Draw (Fig. 3′) BN perpendicular to the natural slope AD; draw BO, making the same angle with BN that E makes with the horizontal, and

then determine the point J so that equation (17) is fulfilled, that is, make AJ a mean proportional between AO and AD; then draw JC parallel to OB. Thus the surface of rupture AC is found, and use can now be made of the relations already deduced in I.

In order to determine J (A, O, and D being given), there are several methods, one of which is indicated in the figure. In all these constructions δ *is assumed.*

Now from equation (13), $E = \frac{1}{2}\gamma\,\overline{JC}^2 \cos(\alpha+\delta)$,

but

$$\frac{CJ}{BO} = \frac{AD-AJ}{AD-AO} = \frac{AD-\sqrt{AD\,.\,AO}}{AD-AO} = \frac{1-\sqrt{\frac{AO}{AD}}}{1-\frac{AO}{D}}.$$

Let $n = \sqrt{\frac{AO}{AD}}$, then $CJ = \frac{1-n}{1-n^2}BO = \frac{BO}{1+n}$.

From Fig. 3,

$$\frac{AO}{AB} = \frac{\sin(\varphi+\delta)}{\cos(\alpha+\delta)}, \qquad \frac{AB}{AD} = \frac{\sin(\varphi-\varepsilon)}{\cos(\alpha-\varepsilon)};$$

and the multiplication of these equations gives

$$n = \sqrt{\frac{\sin(\varphi+\delta)\sin(\varphi-\varepsilon)}{\cos(\alpha+\delta)\cos(\alpha-\varepsilon)}}. \quad . \quad . \quad (18)$$

If $AB = l$, $\qquad BO = \frac{\cos(\varphi-\alpha)}{\cos(\alpha+\delta)}l$;

and by substitution of BO and n in the value for CJ, and of CJ in that for E,

$$E = \left[\frac{\cos(\phi-\alpha)}{n+1}\right]^2 \frac{l^2\gamma}{2\cos(\alpha+\delta)} = \left[\frac{\cos(\phi-\alpha)}{(n+1)\cos\alpha}\right]^2 \frac{h^2\gamma}{2\cos(\alpha+\delta)}. \quad . \quad . \quad (19)$$

For the special case of the earth-surface parallel to the angle of repose, $\varepsilon = \varphi$, $n = 0$, and

$$E = \frac{\cos^2(\varphi-\alpha)}{\cos(\alpha+\delta)} \frac{l^2\gamma}{2} = \left[\frac{\cos(\varphi-\alpha)}{\cos\alpha}\right]^2 \frac{h^2\gamma}{2\cos(\alpha+\delta)}. \quad (20)$$

These formulæ hold good for any value of δ. But the angle δ is determined by equation (3). In order to insert e and r in this formula, the points of application of E and R must be known. The angles δ and ω are connected by the relations in (16*b*), in which there are no other unknown quantities. Since now δ, according to the single assumption of Prof. Weyrauch's theory, is independent of the height, so also is ω, and then for variable h equations (19) and (11) become

$$E = Cl^2, \qquad R = C_1k^2,$$
$$dE = 2Cldl, \qquad dR = 2C_1kdk.$$

Let x and z equal the distance of the point of application of E and R from A, respectively. Now considering the top as the origin or centre of moments,

$$E(l-x) = 2C\int_0^l l^2dl, \qquad R(k-x) = 2C_1\int_0^k k^2dk,$$

and therefore $x = \frac{1}{3}l$ and $z = \frac{1}{3}k$.

Now G must act through the centre of gravity of the ΔABC, and it has been already proved that the points

of application of E and R are at distances $\frac{1}{3}l$ and $\frac{1}{3}k$ respectively above A; hence (Fig. 3′) $ah = ed$ and $hf = g = bd - ah = \frac{1}{3}k \sin \omega - \frac{1}{3}l \sin \alpha$.

Substituting these values in equation (3) and referring to equation (15),

$$AB\,(CJ \cos \delta - AJ \sin \alpha) = AC\,(AC \cos \phi - AJ \sin \omega), \quad . \quad . \quad . \quad (22)$$

or

$$\sin II\,(\sin IV \cos \delta - \sin V \sin \alpha) = \sin III\,(\sin VI \cos \phi - \sin V \sin \omega), \quad (22a)$$

or

$$\cos (\epsilon + \omega)\,[\cos (\phi + \omega) \cos \delta - \sin (\phi + \omega + \alpha + \delta) \sin \alpha]$$
$$= \cos (\alpha - \epsilon)\,[\cos (\alpha + \delta) \cos \phi - \sin (\phi + \omega + \alpha + \delta) \sin \omega]. \quad . \quad . \quad (22b)$$

By means of the two equations (16*b*) and (22*b*) the two unknown quantities δ and ω are completely determined. As soon as these are known, E can be found from equation (19) or (20). Also by the relations in equations (16) and (22), or (16*a*) and (22*b*), the surface of rupture and direction of the earth-pressure may be determined, and can therefore be found by a graphical construction.

III.

HORIZONTAL EARTH-SURFACE.

For this most important practical case it is simply necessary to make $\varepsilon = 0$ in equation (19). The proper values of δ and ω in this case are found from (16*b*) and (22*b*).

Making $\varepsilon = 0$ in equation (22*b*), it becomes

$$\cos\omega\,[\cos(\varphi+\omega)\cos\delta - \sin(\varphi+\omega+\alpha+\delta)\sin\alpha]$$
$$-\cos\alpha\,[\cos(\alpha+\delta)\cos\varphi - \sin(\varphi+\omega+\alpha+\delta)\sin\omega]=0.$$

Since

$$\sin(\varphi+\omega+\alpha+\delta) = \sin(\varphi+\omega)\cos(\alpha+\delta) + \cos(\varphi+\omega)\sin(\alpha+\delta),$$
$$\cos(\alpha+\delta) = \cos\alpha\cos\delta - \sin\alpha\sin\delta,$$
$$\text{and}\qquad \sin(\alpha+\delta) = \sin\alpha\cos\delta + \cos\alpha\sin\delta,$$

the above expression becomes

$$\left.\begin{aligned}
&\cos\omega\cos\delta\cos(\varphi+\omega)\\
&-\cos\omega\sin\alpha\cos\alpha\cos\delta\sin(\varphi+\omega)\\
&\qquad+\cos\omega\sin^2\alpha\sin\delta\sin(\varphi+\omega)\\
&-\cos\omega\sin\alpha\cos\alpha\sin\delta\cos(\varphi+\omega)\\
&\qquad-\cos\omega\sin^2\alpha\cos\delta\cos(\varphi+\omega)\\
&-\cos\alpha\cos\varphi\cos(\alpha+\delta)\\
&+\cos^2\alpha\sin\omega\cos\delta\sin(\varphi+\omega)\\
&\qquad-\cos\alpha\sin\omega\sin\alpha\sin\delta\sin(\varphi+\omega)\\
&+\cos^2\alpha\sin\omega\sin\delta\cos(\varphi+\omega)\\
&\qquad+\cos\alpha\sin\omega\sin\alpha\cos\delta\cos(\varphi+\omega)
\end{aligned}\right\} = 0,$$

which reduces to

$$\left.\begin{aligned}
&\cos\omega\cos(\varphi+\omega)\cos\delta\\
&-\sin\alpha\cos\alpha[\sin(\varphi+\omega)\cos\omega-\cos(\varphi+\omega)\sin\omega]\cos\delta\\
&-\sin\alpha\cos\alpha[\cos(\varphi+\omega)\cos\omega+\sin(\varphi+\omega)\sin\omega]\sin\delta\\
&+[\sin^2\alpha\sin(\varphi+\omega)\cos\omega+\cos^2\alpha\cos(\varphi+\omega)\sin\omega]\sin\delta\\
&+[\cos^2\alpha\sin(\varphi+\omega)\sin\omega-\sin^2\alpha\cos(\varphi+\omega)\cos\omega]\cos\delta\\
&-\cos^2\alpha\cos\varphi\cos\delta+\sin\alpha\cos\alpha\cos\varphi\sin\delta
\end{aligned}\right\}$$
$$=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22c)$$

The expression in the first parenthesis is equal to $\sin\varphi$, in the second to $\cos\varphi$. If in the third $\cos^2\alpha = 1-\sin^2\alpha$, and in the fourth $\sin^2\alpha = 1-\cos^2\alpha$, equation (22*c*) becomes

$$\left.\begin{aligned}
&+\cos\omega\cos(\varphi+\omega)\cos\delta-\sin\alpha\cos\alpha\cos\delta\sin\varphi\\
&\qquad\qquad-\sin\alpha\cos\alpha\sin\delta\cos\varphi\\
&+\sin\delta\sin^2\alpha\sin(\varphi+\omega)\cos\omega+\sin\delta\sin\omega\cos(\varphi+\omega)\\
&\qquad\qquad-\sin^2\alpha\sin\omega\sin\delta\cos(\varphi+\omega)\\
&+\cos\delta\cos^2\alpha\sin(\varphi+\omega)\sin\omega-\cos\delta\cos\omega\cos(\varphi+\omega)\\
&\qquad\qquad+\cos^2\alpha\cos\delta\cos\omega\cos(\varphi+\omega)\\
&-\cos^2\alpha\cos\varphi\cos\delta+\sin\alpha\cos\alpha\cos\varphi\sin\delta
\end{aligned}\right\}=0.$$

Reducing and dividing by $\cos\delta$,

$$\left.\begin{aligned}
&-\sin\alpha\cos\alpha\sin\varphi+\sin^2\alpha\cos\omega\sin(\varphi+\omega)\tan\delta\\
&\qquad\qquad+\sin\omega\cos(\varphi+\omega)\tan\delta\\
&-\sin^2\alpha\sin\omega\cos(\varphi+\omega)\tan\delta\\
&\qquad\qquad+\cos^2\alpha\sin\omega\sin(\varphi+\omega)\\
&+\cos^2\alpha\cos\omega\cos(\varphi+\omega)-\cos^2\alpha\cos\varphi
\end{aligned}\right\}=0.$$

Since

$$\cos\omega\sin(\varphi+\omega)-\sin\omega\cos(\varphi+\omega)=\sin\varphi$$

and

$$\sin \omega \sin (\varphi + \omega) + \cos \omega \cos (\varphi + \omega) = \cos \varphi,$$

this reduces to

$$\begin{aligned} &- \sin \alpha \cos \alpha \sin \varphi + \sin^2 \alpha \sin \varphi \tan \delta \\ &+ \sin \omega \cos (\varphi + \omega) \tan \delta = 0; \end{aligned}$$

and since

$$\cos (\varphi + \omega) \sin \omega = \tfrac{1}{2} \sin (2\omega + \varphi) - \tfrac{1}{2} \sin \varphi,$$

this becomes

$$\tan \delta = \frac{2 \sin \alpha \cos \alpha \sin \varphi}{2 \sin^2 \alpha \sin \varphi + \sin (2\omega + \varphi) - \sin \varphi};$$

and since

$$\sin \alpha \cos \alpha = \tfrac{1}{2} \sin 2\alpha \quad \text{and} \quad 1 - 2 \sin^2 \alpha = \cos 2\alpha,$$

this reduces to

$$\tan \delta = \frac{\sin \varphi \sin 2\alpha}{\sin (2\omega + \varphi) - \sin \varphi \cos 2\alpha}. \quad (23)$$

This equation, therefore, expresses the condition that the "*sum of the moments of E, G, and R is zero.*"

Substituting $\frac{\sin \delta}{\cos \delta}$ for $\tan \delta$ in equation (23), clearing of fractions and factoring,

$$\sin \delta \sin (2\omega + \varphi) - \sin \delta \sin \varphi \cos 2\alpha = \sin \varphi \cos \delta \sin 2\alpha,$$

or

$$\sin\delta\sin(2\omega+\varphi)=\sin\varphi\cos\delta\sin 2\alpha+\sin\varphi\sin\delta\cos 2\alpha.$$

Since $\cos\delta\sin 2\alpha+\sin\delta\cos 2\alpha=\sin(2\alpha+\delta)$,

this becomes

$$\sin\delta\sin(2\omega+\varphi)=\sin\varphi\sin(2\alpha+\delta). \quad . \quad (24)$$

In order to determine ω it is only necessary to make $\varepsilon=0$ in equation (16*b*) express $\sin(\varphi+\omega+\alpha+\delta)$ in terms of sin and cos $(\varphi+\omega)$ and $(\alpha+\delta)$, and then the sin and cos of $(\alpha+\delta)$ in terms of the sin and cos of α and δ. Making $\varepsilon=0$ in equation (16*b*), it becomes

$$\sin(\alpha+\omega)\cos(\alpha+\delta)\cos\omega = \sin(\varphi+\omega+\alpha+\delta)[\cos(\varphi+\omega)\cos\alpha]. \quad (24a)$$

Since

$$\sin(\varphi+\omega+\alpha+\delta)=\sin(\varphi+\omega)\cos(\alpha+\delta)+\cos(\varphi+\omega)\sin(\alpha+\delta)$$
$$\sin(\alpha+\delta)=\sin\alpha\cos\delta+\cos\alpha\sin\delta$$
$$\cos(\alpha+\delta)=\cos\alpha\cos\delta-\sin\alpha\sin\delta;$$

hence

$$\sin(\varphi+\omega+\alpha+\delta)=\sin(\varphi+\omega)\cos\alpha\cos\delta-\sin(\varphi+\omega)\sin\alpha\sin\delta+\cos(\varphi+\omega)\sin\alpha\cos\delta+\cos(\varphi+\omega)\cos\alpha\sin\delta,$$

and equation (24*a*) reduces to

$$\left.\begin{aligned}
&\cos\omega\sin(\alpha+\omega)\cos\alpha\cos\delta \\
&\qquad -\cos\omega\sin(\alpha+\omega)\sin\alpha\sin\delta \\
&-\cos^2\alpha\cos(\varphi+\omega)\sin(\varphi+\omega)\cos\delta \\
&\quad +\cos\alpha\cos(\varphi+\omega)\sin(\varphi+\omega)\sin\alpha\sin\delta \\
&-\cos\alpha\cos^2(\varphi+\omega)\sin\alpha\cos\delta \\
&-\cos^2\alpha\cos^2(\varphi+\omega)\sin\delta
\end{aligned}\right\} = 0. \quad (24b)$$

Dividing by $\cos\delta$,

$$\left.\begin{aligned}
&\cos\alpha\cos\omega\sin(\alpha+\omega) \\
&\qquad -\cos\omega\sin\alpha\sin(\alpha+\omega)\tan\delta \\
&-\cos^2\alpha\cos(\varphi+\omega)\sin(\varphi+\omega) \\
&\quad +\cos\alpha\sin\alpha\cos(\varphi+\omega)\sin(\varphi+\omega)\tan\delta \\
&-\cos\alpha\sin\alpha\cos^2(\varphi+\omega) \\
&-\cos^2\alpha\cos^2(\varphi+\omega)\tan\delta
\end{aligned}\right\} = 0. \quad (24c)$$

Since

$\cos\alpha\cos\omega\sin(\alpha+\omega)$ equals, by expanding $\sin(\alpha+\omega)$, $\sin\alpha\cos\alpha\cos^2\omega+\sin\omega\cos\omega\cos^2\alpha$, and likewise

$$-\cos\omega\sin\alpha\sin(\alpha+\omega)\tan\delta = -\cos^2\omega\sin^2\alpha\tan\delta$$
$$-\cos\alpha\sin\alpha\cos\omega\sin\omega\tan\delta,$$

equation (24*c*) becomes

$$\left.\begin{aligned}
&-\sin\alpha\cos\alpha\left[\cos^2(\varphi+\omega)-\cos^2\omega\right] \\
&-\cos^2\alpha\left[\sin(\varphi+\omega)\cos(\varphi+\omega)-\sin\omega\cos\omega\right] \\
&-\left[\cos^2\alpha\cos^2(\varphi+\omega)+\sin^2\alpha\cos^2\omega\right]\tan\delta \\
&+\sin\alpha\cos\alpha\left[\sin(\varphi+\omega)\cos(\varphi+\omega)\right. \\
&\qquad\qquad \left.-\sin\omega\cos\omega\right]\tan\delta
\end{aligned}\right\} = 0. \quad (24d)$$

Now

$$\cos^2(\varphi+\omega)-\cos^2\omega=\frac{\cos 2(\varphi+\omega)-\cos 2\omega}{2},$$

which equals

$$\frac{2\sin\tfrac{1}{2}[2\omega-(2\varphi+2\omega)]\sin\tfrac{1}{2}[2\omega+(2\varphi+2\omega)]}{2}$$

$$=\frac{2\sin(-\varphi)\sin(2\omega+\varphi)}{2},$$

or $$-\sin(2\omega+\varphi)\sin\varphi,$$

and

$$\sin(\varphi+\omega)\cos(\varphi+\omega)-\sin\omega\cos\omega$$
$$=\tfrac{1}{2}\sin 2(\varphi+\omega)-\tfrac{1}{2}\sin 2\omega;$$

also,

$$\sin\alpha\cos\alpha=\frac{\sin 2\alpha}{2},\text{ and }\cos^2\alpha=\frac{\cos 2\alpha}{2}+\tfrac{1}{2}.$$

Hence, after multiplying by 2, equation (24*d*) reduces to

$$\left.\begin{aligned}
&\sin 2\alpha\sin(2\omega+\varphi)\sin\varphi\\
&\quad-\cos 2\alpha\,\tfrac{1}{2}\sin 2(\varphi+\omega)+\cos 2\alpha\,\tfrac{1}{2}\sin 2\omega\\
&-\tfrac{1}{2}\sin 2(\varphi+\omega)+\tfrac{1}{2}\sin 2\omega\\
&-\tan\delta\cos 2\alpha\cos^2(\varphi+\omega)-\cos^2(\varphi+\omega)\tan\delta\\
&-2\tan\delta\sin^2\alpha\cos^2\omega\\
&\quad+\sin 2\alpha\sin(\varphi+\omega)\cos(\varphi+\omega)\tan\delta\\
&-\sin 2\alpha\sin\omega\cos\omega\tan\delta
\end{aligned}\right\}=0.\quad(24e)$$

Now

$$-2\tan\delta\sin^2\alpha\cos^2\omega = [\text{since } \sin^2\alpha = 1-\cos^2\alpha]$$
$$-[\cos^2\omega - \cos^2\alpha\cos^2\omega]\,2\tan\delta,$$

which equals

$$-\underline{2\cos^2\omega\tan\delta} + 2\tan\delta\cos^2\alpha\cos^2\omega.$$

Also,

$$-\frac{\cos 2\alpha\sin 2(\varphi+\omega)}{2} + \frac{\cos 2\alpha\sin 2\omega}{2}$$
$$= -\cos 2\alpha\left[\frac{\sin 2(\varphi+\omega) - \sin 2\omega}{2}\right]$$
$$= -\frac{\cos 2\alpha[2\sin\varphi\cos(2\omega+\varphi)]}{2}$$
$$= -\underline{\cos 2\alpha\cos(2\omega+\varphi)\sin\varphi},$$

and

$$-\frac{\sin 2(\varphi+\omega)}{2} + \frac{\sin 2\omega}{2} = -\frac{\sin 2(\varphi+\omega) - \sin 2\omega}{2}$$
$$= -\frac{2\sin\tfrac{1}{2}(2\varphi+2\omega-2\omega)\cos\tfrac{1}{2}(2\varphi+2\omega+2\omega)}{2}$$
$$= -\underline{\sin\varphi\cos(2\omega+\varphi)},$$

and

$$-\tan\delta\cos 2\alpha\cos^2(\varphi+\omega) + 2\tan\delta\cos^2\alpha\cos^2\omega$$
$$= \left(\text{by making } \cos^2\alpha = \frac{\cos 2\alpha}{2} + \tfrac{1}{2}\right)$$
$$-\tan\delta\cos 2\alpha[\cos^2(\varphi+\omega) - \cos^2\omega] + \tan\delta\cos^2\omega,$$

or $\underline{\tan\delta\cos 2\alpha\sin(2\omega+\varphi)\sin\varphi} + \tan\delta\cos^2\omega,$

Also,
$$\begin{aligned} &-\cos^2(\varphi+\omega)\tan\delta+\tan\delta\cos^2\omega \\ &=-\tan\delta\,[\cos^2(\varphi+\omega)-\cos^2\omega] \\ &=\underline{\sin\varphi\sin(2\omega+\varphi)\tan\delta.} \end{aligned}$$

Also,

$$\begin{aligned} &\tan\delta\sin 2\alpha\sin(\varphi+\omega)\cos(\varphi+\omega) \\ &-\sin 2\alpha\sin\omega\cos\omega\tan\delta \\ &=\tan\delta\sin 2\alpha\,[\sin(\varphi+\omega)\cos(\varphi+\omega)-\sin\omega\cos\omega] \\ &=\tan\delta\sin 2\alpha\left[\frac{\sin 2(\varphi+\omega)-\sin 2\omega}{2}\right] \\ &=\underline{\tan\delta\sin 2\alpha\sin\varphi\cos(2\omega+\varphi);} \end{aligned}$$

and hence equation (24*e*) becomes

$$\left.\begin{aligned} &+\sin\varphi\,[\sin(2\omega+\varphi)\sin 2\alpha-\cos(2\omega+\varphi)\cos 2\alpha] \\ &\qquad\qquad -\sin\varphi\cos(2\omega+\varphi) \\ &+\sin\varphi\,[\sin(2\omega+\varphi)\cos 2\alpha \\ &\qquad\qquad +\cos(2\omega+\varphi)\sin 2\alpha]\tan\delta \\ &+\sin\varphi\,[\sin(2\omega+\varphi)\tan\delta]-2\cos^2\omega\tan\delta \end{aligned}\right\}=0,(24f)$$

and

$$\tan\delta=\frac{\sin\phi\,[\sin(2\omega+\phi)\sin 2\alpha-\cos(2\omega+\phi)\cos 2\alpha]-\sin\phi\cos(2\omega+\phi)}{2\cos^2\omega-\sin\phi\,[\sin(2\omega+\phi)\cos 2\alpha+\cos(2\omega+\phi)\sin 2\alpha]-\sin\phi\sin(2\omega+\phi)}.$$

By making $\sin 2\alpha=2\sin\alpha\cos\alpha$ and $\cos 2\alpha=1-2\sin^2\alpha$ in the numerator, and $\cos 2\alpha=2\cos\alpha\cos\alpha-1$ and $\sin 2\alpha=2\sin\alpha\cos\alpha$ in the denominator, this becomes

$$\tan\delta=\frac{\sin\phi\,[\sin(2\omega+\phi)2\sin\alpha\cos\alpha-\cos(2\omega+\phi)+\cos(2\omega+\phi)2\sin^2\alpha]-\sin\phi\cos(2\omega+\phi)}{2\cos^2\omega-\sin\phi[\sin(2\omega+\phi)2\cos^2\alpha-\sin(2\omega+\phi)+\cos(2\omega+\phi)2\sin\alpha\cos\alpha]-\sin\phi\sin(2\omega+\phi)},$$

or

$$\tan\delta=\frac{2\sin\phi\sin\alpha\,[\sin(2\omega+\phi)\cos\alpha+\cos(2\omega+\phi)\sin\alpha]-2\sin\phi\cos(2\omega+\phi)}{2\cos^2\omega-2\sin\phi\cos\alpha\,[\sin(2\omega+\phi)\cos\alpha+\cos(2\omega+\phi)\sin\alpha]},$$

which reduces to

$$\tan\delta = \frac{\sin\varphi\sin\alpha\sin(2\omega+\varphi+\alpha)-\sin\varphi\cos(2\omega+\varphi)}{\cos^2\omega-\sin\varphi\cos\alpha\sin(2\omega+\varphi+\alpha)}. \quad (24g)$$

Equating this value of tan δ with that in equation (23),

$$\frac{\sin\varphi\sin\alpha\sin(2\omega+\varphi+\alpha)-\sin\varphi\cos(2\omega+\varphi)}{\cos^2\omega-\sin\varphi\cos\alpha\sin(2\omega+\varphi+\alpha)} = \frac{\sin\varphi\sin 2\alpha}{\sin(2\omega+\varphi)-\sin\varphi\cos 2\alpha}.$$

Dividing by sin φ, clearing of fractions and dividing by sin α, also transposing, this becomes

$$\left.\begin{aligned}&\sin(2\omega+\varphi+\alpha)\sin(2\omega+\varphi)\\&-\sin(2\omega+\varphi+\alpha)\sin\varphi\cos 2\alpha-\frac{\sin 2\alpha}{\sin\alpha}\cos^2\omega\\&+\frac{\sin 2\alpha}{\sin\alpha}\cos\alpha\sin(2\omega+\varphi+\alpha)\sin\varphi\\&-\frac{\cos(2\omega+\varphi)\,[\sin(2\omega+\varphi)-\sin\varphi\cos 2\alpha]}{\sin\alpha}\end{aligned}\right\}=0,$$

or

$$\left.\begin{aligned}&\sin(2\omega+\varphi+\alpha)\sin(2\omega+\varphi)\\&-\sin\varphi\cos 2\alpha\sin(2\omega+\varphi+\alpha)-2\cos\alpha\cos^2\omega\\&+\sin\varphi\,2\cos^2\alpha\sin(2\omega+\varphi+\alpha)\\&-\frac{\cos(2\omega+\varphi)\,[\sin(2\omega+\varphi)-\sin\varphi\cos 2\alpha]}{\sin\alpha}\end{aligned}\right\}=0.$$

Since $$2\cos^2\alpha-\cos 2\alpha = 1,$$

this becomes

$$\sin(2\omega+\varphi+\alpha)\,[\sin(2\omega+\varphi)+\sin\varphi] - 2\cos\alpha\cos^2\omega - D = 0,$$

in which

$$D = \frac{\cos(2\omega+\varphi)\,[\sin(2\omega+\varphi)-\sin\varphi\cos 2\alpha]}{\sin\alpha},$$

or

$$\sin(2\omega+\varphi+\alpha)[2\sin(\omega+\varphi)\cos\omega]-2\cos\alpha\cos^2\omega-D=0,$$

or

$$\sin(2\omega+\varphi+\alpha)\sin(\omega+\varphi)-\cos\alpha\cos\omega-\frac{D}{2\cos\omega}=0. \quad (25)$$

The formulæ for ω, δ, and E can now be found in the simplest manner. Equation (25) is satisfied for $2\omega+\varphi=90°$. Hence,

$$\omega = 45° - \frac{\varphi}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Substituting this value in equation (23), it becomes

$$\tan\delta = \frac{\sin\varphi\sin 2\alpha}{\sin(90-\varphi+\varphi)-\sin\varphi\cos 2\alpha} = \frac{\sin\varphi\sin 2\alpha}{1-\sin\varphi\cos 2\alpha}, \quad . \quad . \quad . \quad . \quad . \quad (27)$$

or the equivalent, but more convenient expression for calculation,

$$\tan(\delta+\alpha)=\frac{\tan\alpha}{\tan^2\left(45°-\frac{\varphi}{2}\right)}. \quad . \quad . \quad . \quad (28)$$

If, finally, $\omega = 45° - \frac{\varphi}{2}$ in equation (10), it becomes, remembering that $k^2 = \frac{h^2}{\cos^2\omega}$,

$$E = \frac{\cos^2\left(\varphi + 45° - \frac{\varphi}{2}\right)}{\cos(\alpha + \delta)} \cdot \frac{h^2\gamma}{2\cos^2\left(45° - \frac{\varphi}{2}\right)}$$

$$= \frac{\cos^2\left(45° + \frac{\varphi}{2}\right)}{\cos^2\left(45° - \frac{\varphi}{2}\right)} \cdot \frac{h^2\gamma}{2\cos(\alpha + \delta)}$$

$$= \frac{\sin^2\left[90° - \left(45° + \frac{\varphi}{2}\right)\right]}{\cos^2\left(45° - \frac{\varphi}{2}\right)} \cdot \frac{h^2\gamma}{2\cos(\alpha + \delta)};$$

hence $$E = \tan^2\left(45° - \frac{\varphi}{2}\right)\frac{h^2\gamma}{2\cos(\alpha + \delta)}, \quad . \quad . \quad . \quad (29)$$

or, from equation (28),

$$E = \frac{\tan\alpha}{\sin(\alpha + \delta)}\,\frac{h^2\gamma}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29a)$$

This last expression, however, when $\alpha = 0$ takes the indeterminate form $\frac{0}{0}$.

The earth-pressure upon a portion of the wall reaching from the depth h_0 to the depth $H = h_0 + h_1$ may be found

from equation (29) by substituting $H^2 - h_0^2$ in place of h^2, as is evident from the following:

Suppose the wall to have a height H, then $E_0 = C_0 \frac{H^2}{2}\gamma$, and likewise for a height h_0

$$E_1 = C_0 \frac{h_0^2}{2}\gamma \therefore E = E_0 - E_1 = C_0 \frac{H^2 - h_0^2}{2}\gamma, \quad . \quad . \quad (29b)$$

C_0 representing the constant quantity.

From equation (29*b*) $E = C(H^2 - h_0^2)$; hence $dE = 2CHdH - 2Ch_0dh_0$. Now let x equal the distance of the centre of pressure below the top of the wall, then

$$Ex = 2C\int_0^H H^2dH - 2C\int_0^h h_0^2dh,$$

or $$C(H^2 - h_0^2)x = \tfrac{2}{3}CH^3 - \tfrac{2}{3}Ch_0^3,$$

or $$x = \frac{2}{3}\frac{H^3 - h_0^3}{H^2 - h_0^2};$$

and if $y =$ the distance from bottom,

$$y = \frac{1}{3}\frac{H^3 - h_0^3}{H^2 - h_0^2}. \quad . \quad . \quad . \quad . \quad (30)$$

Equation (30) holds good when the earth-surface is loaded and the loading is equal to a distributed load of the height h_0. Still, even then, h_0 is often so small that $\frac{h}{3}$ can be substituted for it just as for unloaded earth-surface.

In all cases δ is determined by equation (28).

Instead of using equations (28) and (29), the following simple construction can be used:

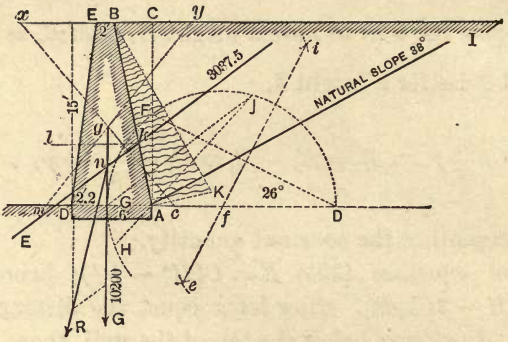

FIG. 4.

Draw (Fig. 4) AC and AD vertically and horizontally, each equal to h, also DF making the angle $FDG = 45° - \frac{\varphi}{2}$ with the horizontal. Through the points D and F describe a circle whose centre lies in AD. Then draw GH parallel to AB, and through A the straight line HJ. Then JG is the direction of the earth-pressure upon the wall AB. If AK is made perpendicular to AB, and equal to AH, then the ΔABK gives the intensity and distribution of the earth-pressure, or

$$E = \gamma \Delta ABK.$$

The proof of this construction is as follows: Conceive, in Fig. 4, JD and FG drawn, then

$$\tan AHG = \frac{AP}{PH} = \frac{AG \cos \alpha}{HG - [AG \sin \alpha = PG]};$$

in which AP represents the perpendicular let fall from A upon GH.

but $$AG : AF :: AF : AD = h,$$

therefore $$AG = \frac{\overline{AF}^2}{h} = h \tan^2\left(45° - \frac{\varphi}{2}\right).$$

Now

$$HG = GD \sin \alpha = (AG + AD) \sin \alpha$$
$$= h \sin \alpha + h \tan^2\left(45° - \frac{\varphi}{2}\right) \sin \alpha;$$

$$\tan AHG = \frac{h \tan^2\left(45° - \frac{\varphi}{2}\right) \cos \alpha}{h \sin \alpha + h \tan^2\left(45° - \frac{\varphi}{2}\right) \sin \alpha - h \tan^2\left(45° - \frac{\varphi}{2}\right) \sin \alpha};$$

therefore

$$\tan AHG = \frac{\cos \alpha}{\sin \alpha} \tan^2\left(45° - \frac{\varphi}{2}\right) = \cot \alpha \tan^2\left(45° - \frac{\varphi}{2}\right).$$

From Fig. 4, $< GDJ = < AHG$, $< GDJ + < JGD = 90°$, and therefore

$$\tan JGD = \cot AHG = \tan \alpha \cot^2\left(45° - \frac{\varphi}{2}\right) = \tan(\alpha + \delta),$$

or $< JGD$ is the angle of the earth-pressure to the horizon.

Since, now, $< AHG = 90° - \alpha - \delta$,

$$AH = \frac{\cos \alpha}{\cos(\alpha + \delta)} AG = h \tan^2\left(45° - \frac{\varphi}{2}\right) \frac{\cos \alpha}{\cos(\alpha + \delta)},$$

and

$$\tfrac{1}{2} AH \,.\, AB = \tan^2\left(45° - \frac{\varphi}{2}\right) \frac{h^2}{2 \cos(\alpha + \delta)} = \frac{E}{\gamma}.$$

For a vertical wall the construction becomes much simpler. Draw, in Fig. 5, $AD = h$ horizontally, then DF making the angle $45° - \frac{\varphi}{2}$ with AD. Draw through D and F a circle with centre in DA and continue it around to K;

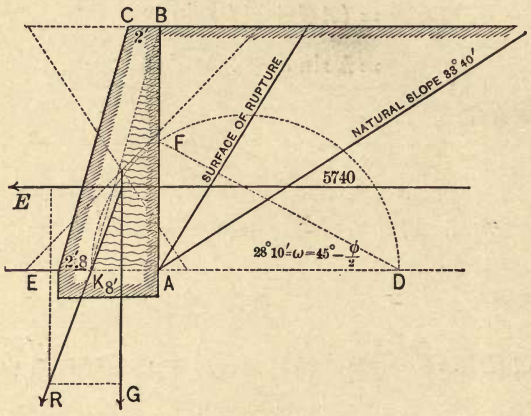

FIG. 5.

then the ΔABK gives the intensity and distribution of the earth-pressure, while in direction it is horizontal.

Hence $$E = \gamma \, \Delta ABK.$$

The proof is as follows (Fig. 5):

$$AK = \frac{AF^2}{AD} = \frac{h^2 \tan^2\left(45° - \frac{\varphi}{2}\right)}{h} = h \tan^2\left(45° - \frac{\varphi}{2}\right)$$

$$\tfrac{1}{2} AB \,.\, AK = \frac{h^2}{2} \tan^2\left(45° - \frac{\varphi}{2}\right) = \frac{E}{\gamma}.$$

As $\alpha = 0$, equation (28) gives $\tan \delta = 0$; $\therefore$ $\delta = 0$ and E act normal to the surface of the wall.

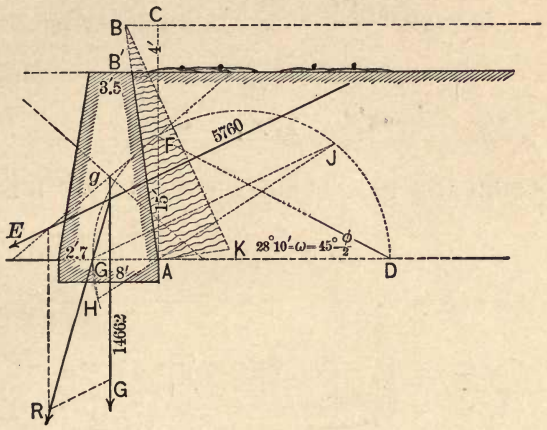

FIG. 6.

Finally, in Fig. 6 is the construction for loaded earth-surface. The point of application of the earth-pressure is always found by drawing through the centre of gravity of ΔABK a parallel to AK and producing it to meet the wall. The proof for this construction is the same as that for Fig. 4.

IV.

EARTH SURFACE PARALLEL TO SURFACE OF REPOSE.

$$\varepsilon = \varphi.$$

For this case,

$$E = \frac{\cos^2(\varphi - \alpha)}{\cos(\alpha + \delta)} \frac{l^2\gamma}{2} = \left[\frac{\cos(\varphi - \alpha)}{\cos\alpha}\right]^2 \frac{h^2\gamma}{2\cos(\alpha + \delta)}; \quad (20)$$

a formula which holds good for all values of δ, and which for $\delta = 0$ or ω gives results usually accepted in previous theories of retaining-walls. In order to find the proper values of δ and ω, equations (16b) and (22b) must be used.

In equation (22b) replace $\sin(\varphi + \omega + \alpha + \delta)$ by $\sin(\varphi + \omega + \alpha)\cos\delta + \cos(\varphi + \omega + \alpha)\sin\delta$, and making $\varepsilon = \varphi$ it becomes

$$\left.\begin{array}{l} +\cos(\varphi+\omega)\cos(\varphi+\omega)\cos\delta \\ \qquad -\cos(\varphi+\omega)\sin(\varphi+\omega+\alpha)\cos\delta\sin\alpha \\ -\cos(\varphi+\omega)\cos(\varphi+\omega+\alpha)\sin\delta\sin\alpha \end{array}\right\} =$$

$$= \left\{\begin{array}{l} +\cos(\alpha-\varphi)\cos(\alpha+\delta)\cos\varphi \\ \qquad -\cos(\alpha-\varphi)\sin(\varphi+\omega+\alpha)\sin\omega\cos\delta \\ -\cos(\alpha-\varphi)\cos(\varphi+\omega+\alpha)\sin\delta\sin\omega; \end{array}\right.$$

dividing by cos δ and transposing,

$$\left.\begin{aligned} &-\frac{\cos(\alpha-\varphi)\cos(\alpha+\delta)\cos\varphi}{\cos\delta} \\ &\qquad +\cos(\alpha-\varphi)\sin(\varphi+\omega+\alpha)\sin\omega \\ &+\cos(\varphi+\omega)\cos(\varphi+\omega) \\ &\qquad -\cos(\varphi+\omega)\sin(\varphi+\omega+\alpha)\sin\alpha \end{aligned}\right\} =$$

$$= \begin{cases} +\cos(\varphi+\omega)\cos(\varphi+\omega+\alpha)\dfrac{\sin\delta}{\cos\delta}\sin\alpha \\ -\cos(\alpha-\varphi)\cos(\varphi+\omega+\alpha)\dfrac{\sin\delta}{\cos\delta}\sin\omega. \end{cases}$$

Since

$$-\frac{\cos(\alpha-\phi)\cos(\alpha+\delta)\cos\phi}{\cos\delta} = -\frac{\cos(\alpha-\phi)\cos\phi\,(\cos\alpha\cos\delta-\sin\alpha\sin\delta)}{\cos\delta}$$

$$= -\cos(\alpha-\phi)\cos\phi\cos\alpha+\cos(\alpha-\phi)\sin\alpha\frac{\sin\delta}{\cos\delta}\cos\phi,$$

the above expression reduces to

$$\tan\delta = \frac{\cos\alpha\cos(\alpha-\phi)\cos\phi-\cos\alpha\cos(\phi+\omega)\cos(\phi+\omega+\alpha)-\cos(\alpha-\phi)\sin\omega\sin(\phi+\omega+\alpha)}{\sin\alpha\cos(\alpha-\phi)\cos\phi-\sin\alpha\cos(\phi+\omega)\cos(\phi+\omega+\alpha)+\cos(\alpha-\phi)\sin\omega\cos(\phi+\omega+\alpha)}$$

and this equation fulfils the condition that the *sum of the moments of G, E, and R shall be zero.*

If equation (16*b*) is treated in a like manner, the resulting equation will fulfil the condition that the *sum of the forces parallel to the surface of rupture shall equal zero.* Making $\varepsilon = \varphi$ in equation (16*b*), it reduces to

$$\sin(\alpha+\omega)\cos(\varphi+\omega)\cos(\alpha+\delta) - \sin(\varphi+\alpha+\omega+\delta)\cos(\varphi+\omega)\cos(\alpha-\varphi) = 0,$$

or

$$\sin(\alpha+\omega)\cos(\alpha+\delta)-\sin(\varphi+\omega+\alpha)\cos(\alpha-\varphi)\cos\delta - \cos(\varphi+\omega+\alpha)\cos(\alpha-\varphi)\sin\delta = 0,$$

or

$$\frac{\sin(\alpha+\omega)\cos\alpha\cos\delta}{\cos\delta} - \frac{\sin(\alpha+\omega)\sin\alpha\sin\delta}{\cos\delta} - \sin(\varphi+\omega+\alpha)\cos(\alpha-\varphi) - \frac{\cos(\varphi+\omega+\alpha)\cos(\alpha-\varphi)\sin\delta}{\cos\delta} = 0;$$

therefore

$$\tan\delta = \frac{\cos\alpha\sin(\alpha+\omega)-\sin(\varphi+\omega+\alpha)\cos(\alpha-\varphi)}{\sin(\alpha+\omega)\sin\alpha+\cos(\varphi+\omega+\alpha)\cos(\alpha-\varphi)}.$$

Setting both values of tang δ equal to each other and clearing of fractions, the following expression is obtained:

$$\begin{aligned}
&+\cos\alpha\cos\varphi\sin\alpha\sin(\omega+\alpha)\cos(\alpha+\varphi)\\
&-\cos\alpha\sin\alpha\sin(\omega+\alpha)\cos(\omega+\alpha)\cos(\omega+\varphi+\alpha)\\
&-\sin\omega\sin\alpha\sin(\omega+\alpha)\cos(\alpha-\varphi)\sin(\varphi+\omega+\alpha)\\
&+\cos\alpha\cos\varphi\cos(\alpha-\varphi)\cos(\varphi+\omega+\alpha)\cos(\alpha-\varphi)\\
&-\cos\alpha\cos(\varphi+\omega)\cos^2(\varphi+\omega+\alpha)\cos(\alpha-\varphi)\\
&-\sin\omega\cos^2(\alpha-\varphi)\sin(\varphi+\omega+\alpha)\cos(\varphi+\omega+\alpha)
\end{aligned}$$

for the first member of the equation, and

$$\begin{aligned}
&+\cos\alpha\cos\varphi\sin\alpha\sin(\omega+\alpha)\cos(\alpha-\varphi)\\
&-\sin\alpha\cos\alpha\sin(\omega+\alpha)\cos(\omega+\alpha)\cos(\varphi+\omega+\alpha)\\
&+\sin\omega\cos\alpha\sin(\omega+\alpha)\cos(\alpha-\varphi)\cos(\varphi+\omega+\alpha)\\
&-\sin\alpha\cos\varphi\cos^2(\alpha-\varphi)\sin(\varphi+\omega+\alpha)\\
&+\sin\alpha\cos(\varphi+\omega)\cos(\varphi+\omega+\alpha)\cos(\alpha-\varphi)\sin(\varphi+\omega+\alpha)\\
&-\sin\omega\cos^2(\alpha-\varphi)\cos(\varphi+\omega+\alpha)\sin(\varphi+\omega+\alpha)
\end{aligned}$$

for the second member.

The first terms, second terms, and sixth terms cancel. Divide the equation by cos $(\alpha - \varphi)$. Terms number 3 combined give

$$-\sin\omega\sin(\omega+\alpha)\,[\sin\alpha\sin(\phi+\omega+\alpha)+\cos\alpha\cos(\phi+\omega+\alpha)],$$

which becomes

$$-\sin\omega\sin(\omega+\alpha)\cos(\varphi+\omega).$$

Terms number 5 combined give

$$-\cos(\phi+\omega)\cos(\phi+\omega+\alpha)\,[\cos\alpha\cos(\phi+\omega+\alpha)+\sin\alpha\sin(\phi+\omega+\alpha)],$$

which becomes

$$-\cos(\varphi+\omega+\alpha)\cos(\varphi+\omega)\cos(\varphi+\omega).$$

Terms number 4 combined give

$$+\cos\varphi\cos(\alpha-\varphi)\,[\cos\alpha\cos(\varphi+\omega+\alpha)+\sin\alpha\sin(\varphi+\omega+\alpha)],$$

which becomes

$$+\cos\varphi\cos(\alpha-\varphi)\cos(\varphi+\omega),$$

and hence, after dividing by cos $(\varphi + \omega)$, the equation above reduces to

$$\cos(\alpha-\varphi)\cos\varphi-\cos(\varphi+\omega+\alpha)\cos(\varphi+\omega)-\sin(\omega+\alpha)\sin\omega=0, \quad (31)$$

and this equation is fulfilled for

$$\omega = 90^\circ - \varphi. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

In order to find that value of δ which satisfies all conditions of equilibrium, substitute the above value of ω in the first expression for tan δ and obtain $\frac{0}{0}$. If, according to

the method for discussing indeterminate fractions, the first differentials of the numerator and denominator and their ratio are found, and ω made equal to $90° - \varphi$, the value of tan δ will be found.

The differential of the numerator is

$$d[-\cos\alpha\cos(\varphi+\omega)\cos(\varphi+\omega+\alpha)-\cos(\alpha-\varphi)\sin\omega\sin(\varphi+\omega+\alpha)],$$

which equals

$$\left.\begin{array}{l} +\cos\alpha\cos(\varphi+\omega+\alpha)\sin(\varphi+\omega) \\ +\cos\alpha\cos(\varphi+\omega)\sin(\varphi+\omega+\alpha) \\ -\cos(\alpha-\varphi)\sin(\varphi+\omega+\alpha)\cos\omega \\ -\cos(\alpha-\varphi)\sin\omega\cos(\varphi+\omega+\alpha) \end{array}\right\} d\omega.$$

Substituting for ω, $90° - \varphi$, this becomes

$$\left.\begin{array}{l} +\cos\alpha\cos(\varphi+90°-\varphi+\alpha)\sin(\varphi+90°-\varphi) \\ +\cos\alpha\cos(\varphi+90°-\varphi)\sin(\varphi+90°-\varphi+\alpha) \\ -\cos(\alpha-\varphi)\sin(\varphi+90°-\varphi+\alpha)\cos(90°-\varphi) \\ +\cos(\alpha-\varphi)\sin(90°-\varphi)\cos(\varphi+90°-\varphi+\alpha) \end{array}\right\} d\omega.$$

As the second term reduces to zero, this becomes

$$[\cos\alpha\sin\alpha-\cos(\alpha-\varphi)\cos\alpha\sin\varphi+\cos(\alpha-\varphi)\cos\varphi\sin\alpha]\,d\omega,$$

or

$$\left[\frac{\sin 2\alpha}{2}-\cos(\alpha-\varphi)(\cos\alpha\sin\varphi-\cos\varphi\sin\alpha)\right]d\omega,$$

or

$$\left[\frac{\sin 2\alpha}{2}-\cos(\alpha-\varphi)\sin(\varphi-\alpha)\right]d\omega$$
$$=\left[\frac{\sin 2\alpha}{2}+\frac{\sin 2(\varphi-\alpha)}{2}\right]d\omega,$$

or

$$\left[\frac{2\sin\frac{1}{2}(2\varphi-2\alpha+2\alpha)\cos\frac{1}{2}(2\varphi-2\alpha-2\alpha)}{2}\right]d\omega,$$

which equals $\sin\varphi\cos(\varphi-2\alpha)\,d\omega$.

The differential of the denominator is

$$\left.\begin{array}{l}+\sin\alpha\cos(\varphi+\omega+\alpha)\sin(\varphi+\omega)\\+\sin\alpha\cos(\varphi+\omega)\sin(\varphi+\omega+\alpha)\\+\cos(\alpha-\varphi)\cos(\varphi+\omega+\alpha)\cos\omega\\+\cos(\alpha-\varphi)\sin\omega\sin(\varphi+\omega+\alpha)\end{array}\right\}d\omega.$$

Substituting $90^\circ-\varphi$ for ω, and this becomes

$$[\sin\alpha\sin\alpha+\cos(\alpha-\varphi)\sin\alpha\sin\varphi+\cos(\alpha-\varphi)\cos\varphi\cos\alpha]\,d\omega,$$

or

$$[\sin^2\alpha+\cos(\alpha-\varphi)\,(\sin\varphi\sin\alpha+\cos\varphi\cos\alpha)]d\omega,$$

or

$$[1-\cos^2\alpha+\cos(\alpha-\varphi)\cos(\alpha-\varphi)]d\omega$$
$$=\left[1-\frac{\cos 2\alpha}{2}-\frac{1}{2}+\frac{\cos 2(\alpha-\varphi)}{2}+\frac{1}{2}\right]d\omega,$$

or $$[1-\sin\varphi\sin(\varphi-2\alpha)]d\omega;$$

therefore

$$\tan\delta=\frac{\sin\varphi\cos(\varphi-2\alpha)}{1-\sin\varphi\sin(\varphi-2\alpha)}. \quad . \quad . \quad (33)$$

To find an expression for the sin δ, clear equation (33)

of fractions and deduce $\tan \delta - \tan \delta \sin \varphi \sin (\varphi - 2\alpha) = \sin \varphi \cos (\varphi - 2\alpha)$. Multiplying by $\cos \delta$,

$$\sin \delta - \sin \delta \sin \varphi \sin (\varphi - 2\alpha) = \sin \varphi \cos (\varphi - 2\alpha) \cos \delta,$$

or

$$\sin \delta = \sin \varphi [\sin \delta \sin (\varphi - 2\alpha) + \cos (\varphi - 2\alpha) \cos \delta];$$

therefore

$$\sin \delta = \sin \varphi \cos (2\alpha - \varphi + \delta), \quad . \quad . \quad (34)$$

from which the results of III. can be deduced.

If the earth-surface is parallel to the surface of repose, or makes the angle φ with the horizontal, then, under the assumption of a plane surface of rupture, $\delta = \varphi$ only when the wall is vertical (make $\alpha = 0$ in equation (33), then $\tan \delta = \tan \varphi$; $\therefore \delta = \varphi$), and $\delta = 0$ only when the angle of the wall with the vertical $\alpha = 45° + \frac{\varphi}{2}$.

As it is often more convenient in determining the direction of the earth-pressure to know the angle $(\alpha + \delta)$ of E with the horizon, $\tan (\alpha + \delta)$ may be expressed in terms of $\tan \alpha$ and $\tan \delta$, remembering that

$$\cos \alpha - \sin \varphi \sin (\varphi - \alpha) = \cos \varphi \cos (\varphi - \alpha),$$

and hence

$$\tan (\alpha + \delta) = \frac{\sin \alpha + \sin \varphi \cos (\varphi - \alpha)}{\cos \varphi \cos (\varphi - \alpha)}. \quad . \quad (34a)$$

With reference to a limited portion of wall which does

not reach as far as the surface, and with reference to loaded earth-surface, the same remarks hold good as in III.

Instead of formulæ (20) and (33) or (34), the following construction may be used.

Draw through A, Fig. 7, a parallel to the earth-surface,

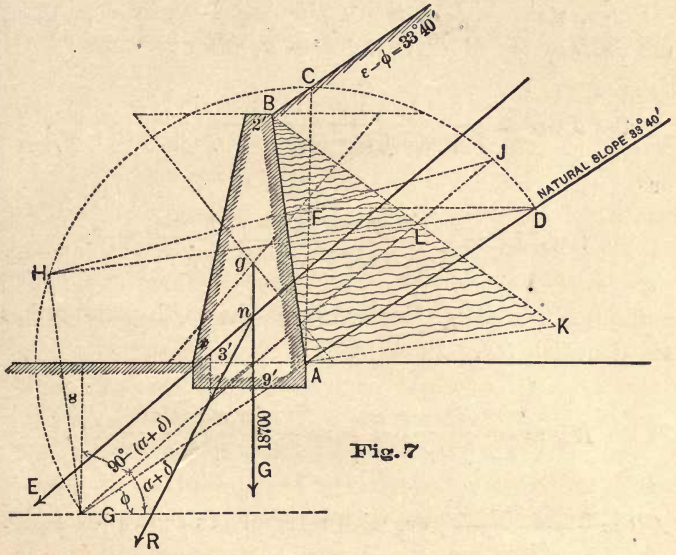

Fig. 7.

Fig. 7.

and with AC as a radius describe the circle ADG. Draw DF horizontal and GH parallel to AB, and then the straight line HFJ. Then the direction of the earth-pressure is GJ; and if AK is made perpendicular to AB and equal to HF, $E = \gamma \Delta ABK$, and the triangle gives the distribution of the pressure. The point of application is found by drawing through the centre of gravity of the triangle a perpendicular to AB.

The proof of this construction is as follows:

Conceive HD drawn, and its intersection with GJ to be at L. Then from the notation of Fig. 3, where $\varepsilon = \varphi$,

$$FD = AD \cos \varphi, \qquad HD = 2AD \cos (\varphi - \alpha).$$

Since, now, $< JLD = < JHD + \varphi - \alpha$, by expressing tan JLD by tan of JHD and $\varphi - \alpha$, after reducing,

$$\tan JLD = \frac{\cos \varphi \cos (2\alpha - \varphi) + \sin 2(\varphi - \alpha)}{1 + \cos^2 (\varphi - \alpha) - \cos \varphi \cos (2\alpha - \varphi)},$$

or

$$\tan JLD = \frac{\sin \varphi \cos (\varphi - 2\alpha)}{1 + \sin \varphi \sin (\varphi - 2\alpha)} = \tan \delta.$$

Since HD is perpendicular to AB, the earth-pressure has the direction GJ. Further,

$$HF = \frac{FD \sin \alpha}{\sin (\alpha + \delta - \varphi)} = \frac{\sin \alpha \cos \varphi}{\sin (\alpha + \delta - \varphi)} AD,$$

$AD = \dfrac{l \cos (\varphi - \alpha)}{\cos \varphi}$, or, with reference to the value of FD.

$\Delta ABK = \dfrac{\cos (\varphi - \alpha) \sin \alpha}{\sin (\alpha + \delta - \varphi)} \dfrac{l}{2}$, and since from equation (34) $\sin (\alpha + \delta - \varphi) \cos (\varphi - \alpha) = \sin \alpha \cos (\alpha + \delta)$,

$$\Delta ABK = \frac{\cos^2 (\varphi - \alpha)}{\cos (\alpha + \delta)} \frac{l^2}{2} = \frac{E}{\gamma}.$$

V.

THE RELIABILITY OF PROF. WEYRAUCH'S THEORY.

PROF. WEYRAUCH'S theory is based upon the single assumption that the surface of rupture is a plane, and is mathematically correct for that assumption. If the surface of rupture is a plane, then all the forces acting upon this plane must be parallel, and can be investigated by considering the equilibrium of the earth-elements, as was done by Rankine (1856), Levy, and Mohr, who proved mathematically that, for earth without cohesion, surface at any inclination and of unlimited extent, the assumption is absolutely correct.

The only question that can be raised, then, is whether it holds good for earth-surface limited, as by a wall.

The latest experiments to determine whether or not the surface of rupture is a plane were made in 1885 by *M. L. Leygue, Ingénieur auxiliaire des Travaux de l'Etat,** who proved to his own satisfaction that the surface was not a plane, but a curve. A brief outline of his experiments will be given, so that any one can form an opinion as to the reliability and worth of the above result or statement.

In one end of a box, with glass sides, having a width of

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* Paper No. 98, *Annales des Ponts et Chaussées*, novembre, 1885 : "Nouvelle Recherche sur la poussée des terres et le profil de revêtement le plus économique."

$0^m.40$ (1.312 ft.), a length of $0^m.80$ (2.624 ft.), and a height of $0^m.73$ (2.395 ft.), he placed a movable board, the board rotating about a knife-edge at the bottom as an axis. The movable board or plane was from $0^m.374$ (1.227 ft.) to $0^m.397$ wide, and from $0^m.20$ (0.651 ft.) to $0^m.25$ (0.820 ft.) high.

This plane was placed in the position to be studied, and fine dry sand filled in behind to the height desired. In the sand were placed horizontal layers of fine plaster. As the layers of plaster would be quite distinct from the sand, any change in their position would be readily detected through the glass sides of the box. Hence, as the plane was rotated, the sand and plaster would change position and a certain amount would break away from the mass in the box, and the line marking the surface of rupture would be clearly defined by the points where the planes of plaster ceased to be horizontal. According to the above, M. Leygue made many experiments: the movable plane taking all positions possible and probable, and the surface of the sand making various angles with the horizontal, and in each and every case the line marking the surface of rupture was found to be a curve passing through the rear toe of the wall and convex to its rear face. And, further, he noticed that this line remained the same after the initial movement of the wall until the movement stopped, when the upper portion of the curve would begin to recede from the wall, and the sand would tend to take its natural slope.

The result of these experiments would apparently overthrow Prof. Weyrauch's theory, or rather make it valueless, as it is founded upon the single assumption that the surface of rupture is a plane. But it appears to the writer,

as it must to any one who stops to consider the matter, that the experiments of M. Leygue were performed upon such a small scale that their results have little real value. The curve is but slightly convex, and, as the experimenter points out no means of ascertaining exactly its equation, but instead introduces a coefficient K, which he calls a coefficient of experience, it is certainly better to follow a single assumption that is, if not exactly true, sufficiently so for all practical purposes.

Experience has shown that the angle of repose for all earths oscillates around the angle 33° within narrow limits, or has a slope of 3 horizontal to 2 vertical.

Thus the slope of rock varies from $\frac{1}{4}$ to $\frac{3}{2}$, and that of dry sand is $\frac{3}{2}$.

Navier observed the general facts that when earth is jarred or exposed to the air or changed in humidity or affected by frost, etc., it changes its qualities, the contiguous parts of the surface successively detach themselves, and the earth tends to take a slope that it would assume if cohesion did not exist, and that this slope approaches $\frac{3}{2}$ but rarely surpasses it. Then it is sufficiently exact to say that earth newly moved and placed behind a rigid wall is in a state analogous to dry sand or the tan φ may be taken as constant and equal to 0.666 or $\varphi = 33° 40'$.

M. Leygue performed many experiments to determine how much the factor cohesion influenced the thrust of the earth, and came to the conclusion that, after the initial movement, the factor cohesion was practically zero. These experiments were, unfortunately, also on a small scale. But it has been generally conceded that cohesion is of but little moment in the deduction of earth thrust, as it is

usually small and very variable, and if omitted the wall will have a little greater stability.

Accordingly, Prof. Weyrauch has omitted cohesion in his theory.

In conclusion, then, it is seen that Prof. Weyrauch's theory is *the theory of to-day*, being founded upon a single assumption, which is, for all practical purposes, sufficiently near the actual fact.

In using the formulæ, the only variable that must be assumed is φ, and that, from what precedes and according to Mr. Trautwine (mentioned farther on), is practically a constant quantity, and is equal to 33° 40′. φ is rarely *less* than 33° 40′, and therefore it is always *safe* to use that value for it.

Prof. Weyrauch's theory will be more fully appreciated and its accuracy, superiority, and simplicity acknowledged after examining the following articles, some of which will be mentioned in Part II.:

Annales des Ponts et Chaussées, mai, 1882 (Paper No. 24): "Note sur la brochure de M. Benjamin Baker, sur la poussée latérale des remblais," par M. J. Curie. A brief review of older theories and a comparison of results as obtained by them and a theory previously advanced by M. Curie. Discussion of examples given by Mr. Baker.

Annales des Ponts et Chaussées, novembre, 1885 (Paper No. 98): "Nouvelle Recherche sur la poussée des terres et le profil de revêtement le plus économique," par M. L. Leygue. A very complete article, giving M. Leygue's experiments to determine the form of the surface of rupture, the thrust of earth, the point of application, etc. Also, a comparison of different theories, old and new.

"The Actual Lateral Pressure of Earthwork," by Benjamin Baker, C.E. (Van Nostrand's Science Series, No. 56). A statement of Mr. Baker's experience on the underground railways of London, and discussions of numerous examples that appear to him to be antagonistic to theory.

"Surcharged and Different Forms of Retaining-walls," by James S. Tate, C.E. (Van Nostrand's Science Series, No. 7). An analytical discussion considering only the overturning stability. Tables of thicknesses.

"Practical Designing of Retaining-walls," by Arthur Jacob, A.B. (Van Nostrand's Science Series, No. 3). An article giving several methods of proportioning walls according to the older theories. Tables of thicknesses.

Van Nostrand's Magazine, Feb., 1882. "Earth-pressure," by Prof. Wm. Cain, C.E.

"Prof. Rankine's Civil Engineering." A complete analytical treatment of the entire subject of retaining-walls.

RECAPITULATION OF FORMULÆ.

Inclined earth-surface, plane:

$$n = \sqrt{\frac{\sin(\varphi + \delta)\sin(\varphi - \varepsilon)}{\cos(\alpha + \delta)\cos(\alpha - \varepsilon)}}. \quad \ldots \quad (18)$$

The tan δ deduced from formulæ (22*b*):

$$\tan\delta = \frac{\sin(2\alpha - \varepsilon) - K\sin 2(\alpha - \varepsilon)}{K - \cos(2\alpha - \varepsilon) + K\cos 2(\alpha - \varepsilon)},$$

in which

$$K = \frac{\cos\varepsilon - \sqrt{\cos^2\varepsilon - \cos^2\varphi}}{\cos^2\varphi},$$

$$E = \left[\frac{\cos(\varphi - \alpha}{(n+1)\cos\alpha}\right]^2 \frac{h^2\gamma}{2\cos(\alpha + \delta)}. \quad \ldots \quad (19)$$

Earth-surface parallel to natural slope:

$$\varepsilon = \varphi;$$

$$E = \left[\frac{\cos(\varphi - \alpha)}{\cos\alpha}\right]^2 \frac{h^2\gamma}{2\cos(\alpha+\delta)}; \quad \ldots \quad (20)$$

$$\omega = 90^\circ - \varphi; \quad \ldots \quad (32)$$

$$\tan(\alpha+\delta) = \frac{\sin\alpha + \sin\varphi\cos(\varphi - \alpha)}{\cos\varphi\cos(\varphi - \alpha)}; \quad \ldots \quad (34a)$$

$$\tan\delta = \frac{\sin\varphi\cos(\varphi - 2\alpha)}{1 - \sin\varphi\sin(\varphi - 2\alpha)}. \quad \ldots \quad (33)$$

Horizontal earth-surface:

$$\omega = 45° - \frac{\varphi}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\tan \delta = \frac{\sin \varphi \sin 2\alpha}{1 - \sin \varphi \cos 2\alpha}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$\tan (\alpha + \delta) = \frac{\tan \alpha}{\tan^2 \left(45° - \frac{\varphi}{2}\right)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$E = \tan^2 \left(45° - \frac{\varphi}{2}\right) \frac{h^2 \gamma}{2 \cos (\alpha + \delta)}; \quad . \quad . \quad . \quad (29)$$

$$E = \frac{\tan \alpha}{\sin (\alpha + \delta)} \cdot \frac{h^2 \gamma}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29a)$$

If $\alpha = 0$, then $\delta = 0$, and

$$E = \tan^2 \left(45° - \frac{\varphi}{2}\right) \frac{h^2 \gamma}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (29d)$$

If $\alpha = \left(45° - \frac{\varphi}{2}\right) = \omega$, then $\delta = \varphi$, and

$$E = \frac{\tan \left(45° - \frac{\varphi}{2}\right)}{\sin \left(45° + \frac{\varphi}{2}\right)} \frac{h^2 \gamma}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (29e)$$

If the surface is loaded, substitute $H^2 + h'^2$ for h^2, or consider h to be the height of the earth increased by the height of an amount of earth weighing as much as the applied load.

PART SECOND.—APPLICATIONS.

EXAMPLES.

BEFORE giving the solutions of the following examples, it may be well to say that earth exerts the greatest pressure *as earth* when it is perfectly dry, or has the minimum angle of repose. Hence φ must be taken at its minimum value in order to obtain satisfactory results from the formulæ. Walls having vertical backs often sustain great pressures due to frost, and, therefore, such walls should have a factor of experience introduced, depending upon the location and structure of the mass to be retained.

A wall is stable when the resultant of all the forces cuts the base, but still it may fail by sliding or bulging. Experience and theory prove that if the resultant cuts the base within the *middle third*, the wall is perfectly stable and will not yield either by sliding or bulging, and also that the wall has a safety factor of at least 2.

The above supposes the wall to be well built and to have a foundation sunk well below the surface of the ground. The foundation should be stepped down from the base of the wall so as to distribute the pressure over more surface.

Weep-holes should always be left in the wall to permit the water to escape from behind. One weep-hole three or four inches wide and the depth of a course of masonry is

generally sufficient to every three or four yards of masonry front of the wall.

When the backing is clean sand, the weep-holes will allow all water to escape; but when it is composed of clay, which retains a great amount of water, a vertical layer of stones or coarse gravel should be placed next to the wall to act as a drain.

The back of the wall should have a batter of at least 1 inch in a foot, in order that the frost may partially spend its force in lifting the earth rather than against the wall.

The masonry should be well bonded together, and no smooth beds allowed.

The resultant of all forces upon the wall should cut the base within the middle third.

Many of these statements will be verified in the following examples.

NOMENCLATURE.

Height of wall	H
Thickness at base	b
Thickness at top	b'
Batter in inches per foot of H on front face	d
Weight per cubic foot	W
Total weight of wall	G
Angle of repose of earth	φ
Angle made by surface of rupture with vertical	ω
Weight of cubic foot of earth	γ
Total thrust of earth against wall	E
Angle made with the horizontal by the surface of the earth	ε
Angle made by rear face of wall with the vertical	α

Angle made with normal by E..... δ
Dist. of point where the resultant pressure cuts the base from the front edge of the wall.. q
The resultant pressure due to E and G..... R

The wall will be considered to be one foot in length.

EXAMPLE 1.—At Northfield, Vt., along the line of the Central Vermont Railroad, is a retaining-wall built of large rough blocks of limestone, $W = 170$ lbs., without cement, which is backed by gravel $\left\{ \begin{matrix} \varepsilon = 0 \\ \gamma = 90 \text{ lbs.} \\ \varphi = 38° \, 0'. \end{matrix} \right\}$ The dimensions are as follows :

$$H = 15 \text{ ft.} \quad b = 6 \text{ ft.} \quad b' = 2 \text{ ft.} \quad d = 1 \text{ in.}$$

This wall was built thirty or more years ago, and seems to be in as good condition now as when first laid.

The wall is several hundred feet in length. Determine the values of E, q, δ.

Solution.—The graphical method of Fig. 4 will be given first. Make $AD = AC = h$; the angle $ADF = \omega = 45° - \frac{\varphi}{2} = 26°$; find the point f by bisecting FD with the perpendicular ei; describe the arc DFG with f as a centre and fD as a radius; draw GH parallel to BA; from H through A draw HJ; then through a point one third the length of AB above A draw E parallel to GJ, and this will give the direction of the thrust, and the angle made with the normal is found to be $\delta = 27° \, 13'$.

Now make AK perpendicular to AB and equal to AH; find the area of the triangle ABK, and multiply the result by $\gamma = 170$, and obtain the value of $E = 3037.5$ pounds.

The next step is to find the centre of gravity of the wall; to do this, make Ex and By equal to six feet; Dm and Ac equal to two feet; connect m with y and x with c, and the point of intersection, g, is the centre of gravity of the cross-section. The value of $G = 15 \times \frac{2+6}{2} \times 170 = 10200$ pounds. Constructing the parallelogram of forces, R is found to cut the base within the middle third or $q = 2.2$ feet; therefore the wall is theoretically safe, so far as overturning is concerned.

In order that no one can say the wall has been favored by taking a large coefficient of friction, it will be taken as 0.4; it is well known that for rough limestone it is nearer 0.75. Then the wall offers a resistance to sliding equal to $10200 \times 0.4 = 4080.$ pounds; as this is much greater than E, it must be still greater than the horizontal component of E, and hence there is no danger from sliding. The plane upon which the wall is supposed to slide is at the bottom of the wall and on *top of the foundation.*

To ascertain the values of E and δ analytically:

Equation (26), $\omega = 45° - \frac{\varphi}{2} = 26°;$

$$\tan \alpha = \frac{2.75}{15} = .18333; \quad \therefore \alpha = 10° 23'.$$

Equation (28), $\tan(\alpha + \delta) = \frac{.18333}{.23788} = .7704;$

$$\therefore \alpha + \delta = 37° 36' \quad \text{and} \quad \delta = 27° 13'.$$

Equation (29a), $E = \frac{.18333}{.61014} \frac{(15)^2 90}{2} = 3037.5$ lbs.

These results are identical with those obtained by the graphical method. To find the value of q the graphical method used above is preferred, as being much simpler than the analytical method.

EXAMPLE 2.—Determine the dimensions of a trapezoidal wall built of dry, rough granite, having a vertical back and being 20 feet high, to safely retain the sides of a sand cut, the surface of the sand being level with the top of the wall. $W = 165$ lbs. $\varphi = 33° 40'$ $\gamma = 100$ lbs.

The graphical solution is given in Fig. 5.

Let AB represent the back face of the wall, 20 ft. in height; make $AD = AB = h$; draw DF, making the angle $ADF = 28° 10' = \omega = 45° - \frac{\varphi}{2}$; pass the arc DFK through D and F, the arc having its centre in AD; draw BK, then the area of $ABK \times \gamma = E = 5740$ lbs. E acts normally to the wall at one third the height.

The dimensions of the wall must now be determined by the process commonly called "cut and try." In this case suppose $b' = 2$ ft. and $b = 8$ ft., and therefore $G = \frac{2+8}{2} \times 20 \times 165 = 16500$ lbs. Now find the centre of gravity g as in example 1, and draw the parallelogram of forces. R is found to cut the base in the middle third, and $q =$ about 2.8 ft. The coefficient of friction of granite on granite is *at least* 0.5, and hence the wall resists sliding at the base by $16500 \times .5 = 8250$ pounds, which is much greater than the thrust E. The triangle ABK represents the intensity of the thrust; and to find the thrust that must be resisted by friction at any height, all one has to

do is to find the area of the triangle above the plane, and multiply it by γ. Thus, it is seen, it is a very simple matter to find whether the wall will slip on any plane above the base.

The above wall, then, is perfectly safe.

Analytical Solution.—$\alpha = 0$, $\omega = 45° - \frac{\varphi}{2}$ = equation (26), which gives $\alpha = 28° 10'$.

Equation (28), $\tan(\alpha + \delta) = 0$; $\therefore$ $\delta = 0$, and E acts normally to the back face of the wall.

Equation (29*d*), $E = .2867 \frac{(20)^2 100}{2} = 5734.$ pounds,

which is the same as obtained graphically within 6 pounds.

The dimensions are obtained by the above graphical method.

From Trautwine's "Engineer's Pocket-book," 1885, p. 690, the above wall would take the proportions $H = 20$ ft., $b = 20 \times .389 = 7.78$ ft., and $b' = 20 \times .096 = 1.92$ ft., and hence $G = 1600$ lbs. The wall to be built of *cut stone.*

For dry rubble, which the example calls for, his proportions are: $H = 20$ ft., $b = 20 \times .528 = 10.56$ ft., $b' = 20 \times .236 = 4.72$ ft., and hence $G = 25212$ lbs., or 8712 lbs., or over 50 cu. ft. more masonry per lineal foot than is necessary. Such walls are no doubt *safe,* but involve a needless waste of material. *Bear in mind, only well-laid walls are considered, and foundations are supposed to be immovable.*

A wall, if built properly, will be so bonded as to leave but few voids, and hence the average weight of the material used in construction may be taken in all practical cases with safety.

We infer, then, that Trautwine's table used above, giving the dimensions of retaining-walls, specifies much larger quantities of material than are absolutely necessary, and leads to waste of material.

EXAMPLE 3.—The same as Example 2 with $\alpha = 8°$, or a batter of 1.68 inches per foot in height. By the graphical method of Fig. 4, assuming $H = 20$, $b = 8$, $b' = 2$, it is found that $E = 6328$ lbs., and that R cuts the base, making $q = 2.7$ ft., or 0.1 ft. less than for a vertical back. The horizontal component of E which tends to make the wall slip is 5200 lbs., which is again less than for the vertical back. Hence it is seen that the wall in this case is less stable, but less liable to slip, and, besides, owing to the inclined back, it will not be affected to so great extent by the action of frost.

The wall, *exclusive of the foundation*, exerts a pressure of less than 14 lbs. per square inch in Ex. 2, and less than 18 lbs. per square inch in Ex. 3 upon the earth.

EXAMPLE 4.—What must be the dimensions of a rubble wall of large blocks of limestone, laid dry, to retain a sand filling which supports two lines of standard-gauge railroad-track?

$H = 15$ ft., $W = 170$ lbs., $\alpha = 8°$, $\varphi = 33° 40'$, $\gamma = 100$ lbs.

Graphical Solution (Fig. 6).—Assuming the moving load of the railroad to be 3000 lbs. per lineal foot, the pressure per square foot as distributed by rails and ties will be about 400 lbs., which will be considered to extend over the whole surface of the fill to compensate for the shocks due to moving trains, etc., or $h = H + 4$, or 19 ft.

Make $AC = H + 4$; make $AD = AC$; draw FD, making

an angle of 28° 10′ $= \omega = 45° - \frac{\varphi}{2}$ with AD; describe the arc $DJFGH$, the centre being in AD; draw GH parallel to AB, then HJ is the direction of the earth-thrust E, which acts at a point above A equal to one third AB; draw AK perpendicular to AB and equal to AH, then the triangle $BAK \times \gamma$ gives the intensity of E or 5760 lbs.

It is evident that the wall could not be triangular, since the thrust at the top is not zero, as the triangle ABK indicates. To find the least allowable value of b', find the thrust at a distance of, say, 1 ft. below the top: this is about 375 lbs., or horizontally about 300 lbs. Then, in order that the top stone will not slip off, it must have a width of $\frac{300}{170 \times 0.5} = 3.53$ ft. (0.5 = coefficient of friction). Hence b' may safely equal 3.5 ft.

The resultant R will cut the base within the middle third, if $b = 8$ ft. and q will equal 2.7 ft.

Analytical Solution.

Equation (26), $\omega = 45° - \frac{\varphi}{2} = 28° 10'.$

Equation (28), $\tan(\alpha + \delta) = \frac{.14054}{.28670} = .4901;$

$\therefore 26° 7' = (\alpha + \delta)$ and $\delta = 18° 7'.$

Equation (29a), $E = \frac{.14054}{.44020} \frac{(19)^2}{2} \frac{100}{} = 5758$ lbs.

Dimensions $b' = 3.5$ ft., $b = 8$ ft., obtained by graphical method above.

If the above wall had a vertical back and was *not* loaded by the railroad, Mr. Trautwine's proportions would be (p. 690, 1885) about $H = 15$ ft., $b = 15 \times 0.51 = 7.65$ ft., $b' = 15 \times 0.343 = 5.15$ ft., and $G = 16320$ lbs.; and hence the wall contains a few more cu. ft. per foot run, and is also more stable (as a vertical-back wall is more stable than one with a battered back, dimensions being the same), but the additional weight of the railroad has not been considered at all.

EXAMPLE 5.—What must be the dimensions of a 20-ft. wall, to retain the foot of a side of a deep sand cut; material to be rough blocks of limestone (case of surcharge)?

$W = 170$ lbs.; $H = 20$ ft.; $\gamma = 100$ lbs.
$\alpha = 8°$; $\varepsilon = \varphi = 33° 40'$;

Graphical Solution (Fig. 7).—Let AB represent the back face of the wall, BC the natural slope and surface of the cut; draw GAD parallel to the surface BC, and, with A as a centre and AC as a radius, describe the arc $DCHG$; draw GH parallel to AB; draw DF horizontal, and through H and F draw HJ; then JG is the direction of E.

Make AK perpendicular to AB and equal to HF, then the area of $ABK \times \gamma = E = 23600$. Assuming $b' = 2$ ft. and $b = 9$ ft., R is found to cut the base within the middle third, or $q = 3$ ft.

Analytical Solution.

Equation (32), $\omega = 90° - \varphi = 56° 20'$.

Equation (34*a*),

$$\tan(\alpha + \delta) = \frac{.13917 + .55436 \times .90133}{.83228 \times .90133} = .8517;$$

$$\therefore \alpha + \delta = 40° 25' \text{ and } \delta = 32° 25'.$$

Equation (20), $$= \frac{.90133}{.99027} \frac{(20)^2 100}{2 \times .76135} = E = 23600 \text{ lbs.}$$

The wall is proportioned by graphical method above.

These five examples illustrate the method of using the graphical constructions and the formulæ. The graphical method seems to be the easier, and is fully accurate enough for all practical purposes.

It has been noticed, no doubt, that φ has been taken throughout, excepting in the first example, to equal 33° 40′, or equivalent to a slope of about 1½ to 1. Mr. Trautwine says, on p. 690 of his "Engineer's Pocket-book," 1885: "*For practical purposes, we may say that dry sand, gravel, and earths slope at* 30° 41′ *or* 1½ *to* 1, *as abundant experience on railroad embankments proves.*" This statement is reasonable, and for the majority of earths the angle is too small; hence walls proportioned for $\varphi = 33° 40'$ will be on the safe side.

In all that precedes it is supposed that there is no friction between the earth and the wall, or, in other words, δ does not depend upon the structure of the wall for its value in any respect.

Now, it is plain that as soon as any movement of the wall takes place the friction existing between the wall and the earth has been overcome; or if a coating of earth sticks to the wall, as is usual, the friction overcome is that of earth on earth; if φ' represents the coefficient of friction of earth and walls, then the direction of E must make an angle with the normal to the back face of the wall equal to at least φ'. To introduce φ' into Professor Weyrauch's theory it is only necessary to find the value of δ as given by his formulæ, and see if it is greater or less than φ'; if it is less, use the value of φ' to determine the *direction* of E; if greater, use its value and omit φ' altogether. In case the earth sticks to the wall it is evident that $\varphi' = \varphi$. Bear in mind φ' does not affect the value of E in any respect. *In finding the value of E, δ as given by the formulæ must be used.* To repeat, φ' *merely affects the direction of E, not its value.*

If φ' is considered equal to zero, rough rubble walls certainly, if vertical on the back, have a large factor of safety if proportioned by Prof. Weyrauch's formulæ. In fact, it appears to the writer that φ' may be introduced into the theory, and then leave the wall with a factor of safety of nearly 2, provided R cuts the base within the middle third. Unless the back of the wall is exceedingly smooth, φ' may be taken equal to φ; and when the wall *is* very smooth it is better to take φ' equal to zero and let δ retain the value as deduced from the formulæ. If φ' is found to be less than δ, use the value of δ; if greater than δ, use the value of φ'. If the factor φ' be introduced into the above examples, the value of q will be found to be materially increased, and hence the amount of masonry may be made

less. These different values are easily obtained ; instead of considering E to make the angle δ with the normal, consider it as making the angle φ', e.g. if $\delta < \varphi'$; and proceed in precisely the same manner as if using δ instead of φ'.

Mr. Benjamin Baker, in "The Actual Lateral Pressure of Earthwork," *Van Nostrand's Science Series*, No. 56, cites many cases that to him seem positive proof that no reliance can be placed upon the values of E as deduced by theory. For instance, he saw a wall made of wooden paving-blocks retaining a pile of macadam screenings:

$H = 4$ ft., $b = 1$ ft., $b' = 1$ ft., $W = 46$ lbs., $\varepsilon = 0$, $h = 3.75$ ft., $\varphi = 1.2$ ft. horizontal to 1 vertical $= 39° 48'$, $\gamma = 101$.

This wall, according to Coulomb's and also Weyrauch's theory, would be overturned. But in reality the wall was not overturned, but stood, and hence Mr. Baker's doubt in theories as being correct and practical ; but now introduce for δ $\varphi' = \varphi = 39° 48'$, hence $q = 0.16$ ft., and the resultant R cuts the base, and therefore the wall, by theory, ought to stand ; and the example, instead of being against theory, is in reality in direct agreement.

Another example given by Mr. Baker. Lieut. Hope experimented with a wall built of brick laid in wet sand:

$H = 10$; $\quad \varepsilon = 0$;
$b = b' = 1.92$; $\quad h = 10$;
$W = 100$; $\quad \gamma = 95.5$;
$\varphi = 36° 53'$.

When the backing had reached a height of 8 ft., the top leaned about 1½ inches ; and when the backing had reached the top, the wall leaned about 4 inches, and then fell.

According to Coulomb's and also Weyrauch's theory, the wall would have fallen much sooner than it did, thus giving an apparent disagreement between fact and theory.

Now introduce the factor $\varphi' = \varphi = 36° 53'$, and, deducing q, it will be found to be negative, considering the wall as leaning 4 inches, and equal to zero for the wall vertical.

Hence, here is another case where practice and theory agree.

Mr. Baker cites many other instances which seem to depart from theory, but, in reality, like the above two examples, they only tend to give the engineer greater faith in formulæ used properly.

Any one desiring to pursue the discussion of Mr. Baker's examples will find them clearly explained by Prof. Wm. Cain, C.E., in "Earth Pressure," *Van Nostrand's Magazine,* February, 1882.

Mr. Baker says: "If an engineer could tell by inspection the supporting power and frictional adhesion of every bit of soil laid bare, or see through 5 or 10 feet of earth into a 'pot hole' or layer of slimy silt, he might avoid many failures, and even hope to frame some useful equations for obtaining the required thickness of a dock wall. Taking things as they are, however, it is hardly worth while to use even a scale and compass in such work, for, being in possession of all the information obtainable about the foundation and backing, an engineer may at once sketch as suitable a cross-section for the particular case as he could hope to arrive at after any amount of mathematical investigation. Something must be assumed in any event, and it is far more simple and direct to assume at once the thickness of the wall than to derive the latter from equations based

upon a number of uncertain assumptions as to the bearing power of the foundations, the resistance to gliding, and other elements. This being so, it has often struck the author that the numerous published tables giving the calculated required thicknesses of retaining-walls to three places of decimals stand really on exactly the same scientific basis, and have the same practical value, as the weather forecasts for the year in Old Moore's Almanack."

Perhaps Mr. Baker is capable of "assuming at once" the thickness of a retaining-wall and the economical cross-section, but it is wholly due to the fact of having had many

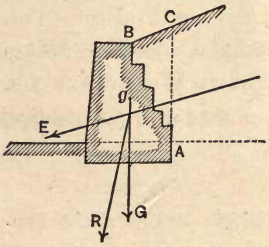

FIG. 8.

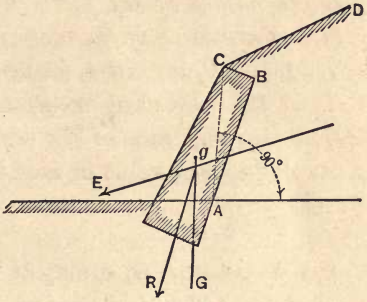

FIG. 9.

years of experience on the underground railways of London. When an engineer can look over the location of his retaining-wall proposed, and mentally, from previous experience, decide upon the angle of repose, φ, the values of E and δ resulting, then he may assume at once the values of b and b'; but *then* the whole thing from beginning to end is an assumption, or rather many assumptions. Is it not better to make a *few* of these assumptions and then deduce mathematically the values of E, δ, b, etc.? Such, certainly, is the better way for the engineer who has

not had very wide experience. As to tables of thicknesses, they are, as Mr. Baker says, of little practical value, excepting, perhaps, those relating to rectangular walls and a level earth-surface. It has already been pointed out that Mr. Trautwine's tables give an amount of material in excess of that usually necessary.

For the benefit of those who wish to use walls stepped on the rear face or inclined backward, Prof. Rankine's * method of treatment is given, as being the best at the present time.

Fig. 8.—Let AB represent a wall, and draw the vertical line AC; then G is made up of *the weight of the wall and also the weight of the earth ACB, and acts through the centre of gravity g of the compound mass situated in front of the line AC, or, better, plane AC.*

Thus far, according to Prof. Rankine. Now consider AC as the rear face of the wall, and by Weyrauch's formulæ deduce the value of E and δ; E acts at a point equal to $\frac{AC}{3}$ above A.

Fig. 9.—Again, according to Rankine, conceive the vertical plane AC. If the prism ABC consisted of earth, it would be supported by the earth behind it, and hence it can be said that the earth exerts an upward pressure against the prism just equal to the weight of a prism of earth equal to ABC; therefore G is equal to the *weight of the masonry less the weight of a prism of earth equal to ABC,* and acts through the centre of gravity of the compound mass, the prism ABC *having a weight equal to the weight of the masonry therein less the weight of an equal prism of earth.*

* "Rankine's Civil Engineering," p. 402.

Now, applying Weyrauch's method, use AC as the back face of the wall, and proceed as for Fig. 8.

Weyrauch's formulæ hold good for fluid pressure: $\varepsilon = 0$, $\varphi = 0$, $\delta = 0$, and hence equation (29*a*) becomes

$$E = \sec \alpha \frac{h^2 \gamma}{2}, \text{ or, for a vertical wall,}$$

$$E = \frac{h^2 \gamma}{2}, \text{ both being well-known formulæ.}$$

VALUES OF *W*.

Name of Substance.	Crushing Lds. in tons per sq. ft.	ϕ'.	Average lbs. per cu. ft.
Alabaster	.....	The coefficients depend upon the nature of the surfaces, whether rough, smooth, etc.	144
Brick, best pressed	40 to 300		150
" common hard	.....		125
" soft inferior	.....		100
Chalk	20 to 30		156
Cement, loose	.....		49.6 to 102
Flint	.....		162
Felspar	.....		166
Granite	300 to 1200		170
Gneiss	.....		168
Greenstone, trap	.....		187
Hornblende, black	.....		203
Limestones and Marbles, ordinary	250 to 1000		164.4 to 168
Mortar, hardened	.....		103
Quartz, common	.....		165
Sandstone	150 to 550		151
Shales	.....		162
Slate	400 to 800		175
Soapstone	.....		170

VALUES OF γ AND φ.

Name of Substance.	Angle of Repose.	Average lbs. per cu. ft.
Earth, common loam, loose	Practically 33° 40′	72 — 80
" " " shaken		82 — 92
" " " rammed moderately		90 — 100
Gravel		90 — 106
Sand		90 — 106

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